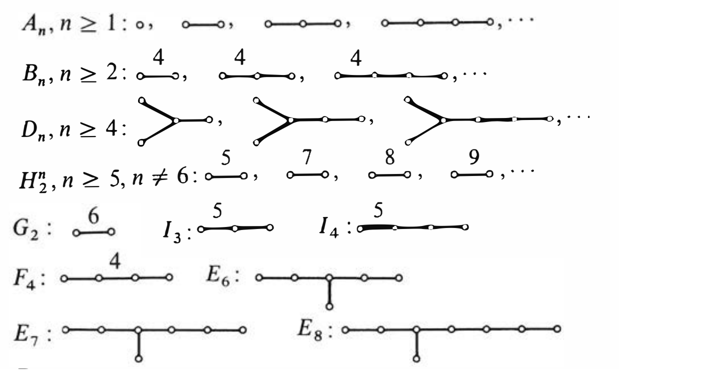
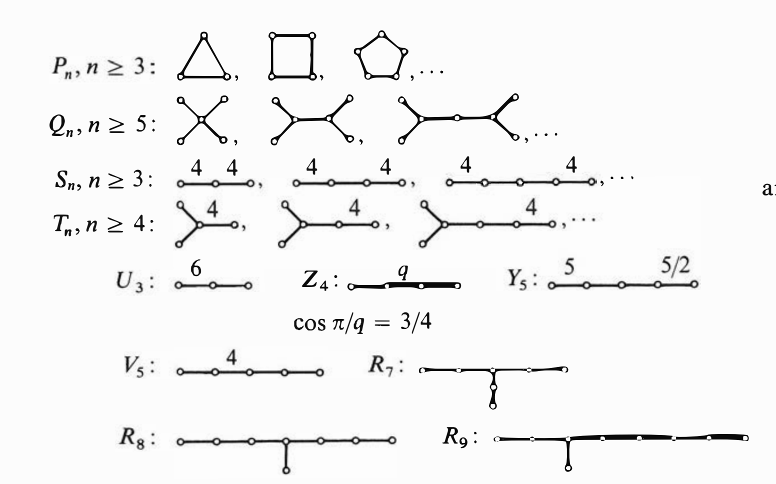
Labelled Graphs, Determinants and Dynkin Diagrams

Source:Benson, Grove, Finite Reflection Groups





$$\det A_n = \det A_{n-1} - 1/4 \det A_{n-2}$$
$$= n/2^{n-1} - 1/4(n-1)/2^{n-2}$$
$$= (n+1)/2^n > 0.$$
$$\det B_n = \det A_{n-1} - (\sqrt{2}/2)^2 \det A_{n-2}$$
$$= n/2^{n-1} - (n-1)/2^{n-1} = 1/2^{n-1} > 0,$$
$$\det D_n = \det A_{n-1} - 1/4 \det A_{n-3}$$
$$= n/2^{n-1} - (n-2)/2^{n-1} = 1/2^{n-2} > 0,$$
$$\det I_3 = \det A_2 - \alpha^2 \det A_1$$
$$= 3/4 - \alpha^2 = (3 - \sqrt{5})/8 > 0,$$
$$\det I_4 = 1/2 - 3\alpha^2/4 = (7 - 3\sqrt{5})/32 > 0,$$
$$\det F_4 = \det B_3 - (1/4) \det A_2 = 1/16 > 0,$$

and for $n = 6, 7, 8$

$$\det E_n = \det D_{n-1} - 1/4 \det A_{n-2}$$
$$= 1/2^{n-3} - (n-1)/2^n = (9-n)/2^n > 0.$$
$$\det P_n = 0 \text{ (row sum 0)}$$
$$\det Q_n = \det D_{n-1} - 1/4 \det D_{n-3} = 0,$$
$$\det S_n = \det B_{n-1} - 1/2 \det B_{n-2} = 0,$$
$$\det T_n = \det B_{n-1} - 1/4 \det B_{n-3} = 0,$$
$$\det U_3 = \det A_2 - 3/4 \det A_1 = 0,$$
$$\det Y_5 = \det I_4 - \beta^2 \det I_3$$
$$= 1/2 - 3\alpha^2/4 - \beta^2(3/4 - \alpha^2)$$
$$= 3(1 - 2\alpha + 2\beta)/16 = 0,$$
$$\det V_5 = \det B_4 - 1/4 \det A_3 = 0,$$
$$\det R_7 = \det E_6 - 1/4 \det A_5 = 0,$$
$$\det R_8 = \det E_7 - 1/4 \det D_6 = 0,$$
$$\det R_9 = \det E_8 - 1/4 \det E_7 = 0.$$

Graph	Base
A_n	$r_i = e_{i+1} - e_i, 1 \le i \le n.$
B_n	$r_1 = e_1, r_i = e_i - e_{i-1}, 2 \le i \le n.$
D_n	$r_1 = e_1 + e_2, r_i = e_i - e_{i-1}, 2 \le i \le n.$
H_2^n	$r_1 = (1, 0), r_2 = (-\cos \pi/n, \sin \pi/n).$
G_2	$r_1 = e_2 - e_1, r_2 = e_1 - 2e_2 + e_3.$
I_3	$r_1 = \beta(2\alpha+1, 1, -2\alpha), r_2 = \beta(-2\alpha-1, 1, 2\alpha),$ $r_3 = \beta(2\alpha, -2\alpha-1, 1).$
I_4	$r_1 = \beta(2\alpha+1, 1, -2\alpha, 0), r_2 = \beta(-2\alpha-1, 1, 2\alpha, 0),$ $r_3 = \beta(2\alpha, -2\alpha-1, 1, 0), r_4 = \beta(-2\alpha, 0, -2\alpha-1, 1).$
F_4	$r_1 = -(1/2)\Sigma_1^4 e_i, r_2 = e_1, r_3 = e_2 - e_1, r_4 = e_3 - e_2.$
E_6	$r_1 = (1/2)(\Sigma_1^3 e_i - \Sigma_4^8 e_i), r_i = e_i - e_{i-1}, 2 \le i \le 6.$
E_7	$r_1 = (1/2)(\Sigma_1^3 e_i - \Sigma_4^8 e_i), r_i = e_i - e_{i-1}, 2 \le i \le 7.$
E_8	$r_1 = (1/2)(\Sigma_1^3 e_i - \Sigma_4^8 e_i), r_i = e_i - e_{i-1}, 2 \le i \le 8.$

Base	Group	$\lvert\Delta\rvert$	Root system Δ
A_n	$\mathscr{A}_n$	$n^2 + n$	$\pm(e_i - e_j), 1 \le j < i \le n+1.$
B_n	$\mathscr{B}_n$	$2n^2$	$\pm e_i, 1 \le i \le n; \pm e_i \pm e_j, 1 \le j < i \le n.$
D_n	$\mathscr{D}_n$	$2n(n-1)$	$\pm e_i \pm e_j, 1 \le j < i \le n.$
H_2^n	$\mathscr{H}_2^n$	$2n$	$(\cos j\pi/n, \sin j\pi/n), 0 \le j \le 2n-1.$
G_2	$\mathscr{G}_2$	12	$\pm(e_i - e_j), 1 \le j < i \le 3; \pm(1, -2, 1),$ $\pm(-2, 1, 1), \pm(1, 1, -2).$
I_3	$\mathscr{I}_3$	30	$\pm e_i, 1 \le i \le 3; \beta(\pm(2\alpha+1), \pm 1, \pm 2\alpha)$, and all even permutations of coordinates.
I_4	$\mathscr{I}_4$	120	$\pm e_i, 1 \le i \le 4; (1/2)(\pm 1, \pm 1, \pm 1, \pm 1);$ $\beta(\pm 2\alpha, 0, \pm(2\alpha+1), \pm 1)$, and all even permutations of coordinates.
F_4	$\mathscr{F}_4$	48	$\pm e_i, 1 \le i \le 4; \pm e_i \pm e_j, 1 \le j < i \le 4;$ $(1/2)\Sigma_1^4 \varepsilon_i e_i, \varepsilon_i = \pm 1.$
E_8	$\mathscr{E}_8$	240	$\pm e_i \pm e_j, 1 \le j < i \le 8; (1/2)\Sigma_1^8 \varepsilon_i e_i,$ $\varepsilon_i = \pm 1, \Pi_1^8 \varepsilon_i = -1.$
E_7	$\mathscr{E}_7$	126	Those roots of $\mathscr{E}_8$ orthogonal to $u = (1/2)(1, 1, 1, 1, 1, 1, 1, -1).$
E_6	$\mathscr{E}_6$	72	Those roots of $\mathscr{E}_7$ orthogonal to $r_8 = e_8 - e_7.$

$\mathscr{G}$	$\lvert\mathscr{G}\rvert$	$\mathscr{G}$	$\lvert\mathscr{G}\rvert$
$\mathscr{A}_n$	$(n+1)!$	$\mathscr{I}_3$	$2^3 \cdot 3 \cdot 5$
$\mathscr{B}_n$	$2^n \cdot n!$	$\mathscr{I}_4$	$2^6 \cdot 3^2 \cdot 5^2$
$\mathscr{D}_n$	$2^{n-1} \cdot n!$	$\mathscr{E}_6$	$2^7 \cdot 3^4 \cdot 5$
$\mathscr{H}_2^n$	$2n$	$\mathscr{E}_7$	$2^{10} \cdot 3^4 \cdot 5 \cdot 7$
$\mathscr{G}_2$	12	$\mathscr{E}_8$	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$
$\mathscr{F}_4$	$2^7 \cdot 3^2$		