Coordinates of a Vector

If \( B = \{ b_1, \ldots, b_n \} \) is a basis of the vector space \( V \), then every element \( x \in V \) can be written uniquely as \( x = a_1 b_1 + \cdots + a_n b_n \). We write these coefficients in a column vector, which we call the coordinate vector of \( x \) with respect to \( B \):

\[
[x]_B = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}
\]

We also define the reverse operation: Given a coordinate vector, we consider the vector given with respect to a certain basis by these coefficients:

\[
B \ast \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} := a_1 b_1 + \cdots + a_n b_n.
\]

Be aware that, while the coordinate vector \([x]_B\) represents \( x \) it is not equal to \( x \).
We note that for vectors \( x, y \in V \) and \( \alpha \in \mathbb{F} \), we have that

\[
[x + y]_B = [x]_B + [y]_B \quad \text{and} \quad \alpha \cdot [x]_B = [\alpha \cdot x]_B.
\]

Thus we can perform calculations in \( V \) with coefficient vectors instead of the original vectors.

Changing Vector Coordinates

It is important to understand that the same vector can have different coordinate vectors with respect to different bases. Vice versa, the same coordinate vector will represent different vectors with respect to different bases.

Let \( B = \{ b_1, \ldots, b_n \} \) and \( C = \{ c_1, \ldots, c_n \} \) be bases of \( V \). Then we can express any basis vector of \( B \) in terms of \( C \). Suppose that \( b_j = \sum_{i=1}^{n} a_{i,j} c_i \). Then, if \( v \in V \) such that \([v]_B = (d_1, \ldots, d_n)^T\) we have that

\[
y = \sum_{j=1}^{n} d_j b_j = \sum_{j=1}^{n} d_j \sum_{i=1}^{n} a_{i,j} c_i = \sum_{i=1}^{n} \left( \sum_{j=1}^{n} a_{i,j} d_j \right) c_i.
\]

Thus the \( i \)-th coefficient of \([y]_C\) is \( \sum_{j=1}^{n} a_{i,j} d_j \).

This last formula looks like the formula for multiplication of a matrix \((a_{i,j})_{i,j}\) with a vector \((d_1, \ldots, d_n)^T\). Thus if we set

\[
e_{[id]}_B = (a_{i,j})_{i,j} = \left( \begin{array}{c|c|c} b_1 \mid b_2 \mid \cdots \mid b_n \end{array} \right)_C
\]

we see that

\[
[y]_C = e_{[id]}_B [y]_B.
\]

The general rule is that we read bases from right to left (in the same way as we apply transformations) as to \([id]_B \) from and that when multiplying both factors must have the same basis in common.
This formula generalizes easily to multiple base changes. If $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are bases we have that
\[
\mathcal{D} \{ \text{id} \}_\mathcal{B} \cdot \mathcal{Y}_\mathcal{B} = \mathcal{Y}_\mathcal{D} = \mathcal{D} \{ \text{id} \}_\mathcal{C} \cdot \mathcal{Y}_\mathcal{C} = \mathcal{D} \{ \text{id} \}_\mathcal{C} \cdot (\mathcal{C} \{ \text{id} \}_\mathcal{B} \cdot \mathcal{Y}_\mathcal{B}) = (\mathcal{D} \{ \text{id} \}_\mathcal{C} \cdot \mathcal{id} \}_\mathcal{B} \cdot \mathcal{Y}_\mathcal{B}
\]
As we can have any column vector for $\mathcal{Y}_\mathcal{B}$, we get that
\[
\mathcal{D} \{ \text{id} \}_\mathcal{B} = \mathcal{D} \{ \text{id} \}_\mathcal{C} \cdot \mathcal{id} \}_\mathcal{B}
\]
Clearly $\mathcal{B} \{ \text{id} \}_\mathcal{B}$ is the identity matrix. Thus we get that $\mathcal{D} \{ \text{id} \}_\mathcal{B}$ is invertible and
\[
\mathcal{D} \{ \text{id} \}_\mathcal{B}^{-1} = \mathcal{D} \{ \text{id} \}_\mathcal{C}
\]

Matrices for linear transformations

The situation with matrices for linear transformations is rather similar:
Suppose that $T : U \rightarrow V$ is a linear transformation and that $\mathcal{B} = \{ b_1, \ldots, b_n \}$ is a basis of $U$ and $\mathcal{C} = \{ c_1, \ldots, c_m \}$ a basis of $V$.
If $T(b_j) = a_{1j} + a_{2j}c_2 + \cdots + a_{mj}c_m$, then $[T(b_j)]_\mathcal{C} = (a_{1j}, a_{2j}, \ldots, a_{mj})^T$. We thus set
\[
[T]_\mathcal{B} = \begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\]
and get that for each $\mathbf{x} \in U$ the action of $T$ is given by matrix multiplication with the coordinate vector
\[
[T(\mathbf{x})]_\mathcal{C} = [T]_\mathcal{B} \cdot [\mathbf{x}]_\mathcal{B}
\]
(The situation of a base change thus is a special case for the linear transformation $\text{id}$ which maps every vector to itself.)
If $T : U \rightarrow U$ and only one basis is involved, we might use $[T]_\mathcal{B}$ to denote the (necessarily quadratic) coordinate matrix $\mathcal{B} \{ [T] \}_\mathcal{B}$.

Describing Linear Transformations by matrices

Let $U$ be an $n$-dimensional vector space and $V$ an $m$-dimensional vector space. Suppose that $A \in \mathbb{F}^{m \times n}$. For every choice of bases $\mathcal{B}$ of $U$ and $\mathcal{C}$ of $V$ we can interpret $A$ as matrix for a linear transformation $T : U \rightarrow V$, such that $T(\mathbf{x}) = \mathcal{C} \cdot (A \cdot [\mathbf{x}]_\mathcal{B})$.
(A different choice of bases will yield different transformations.)

Base change for linear transformation

If $T$ is a linear transformation given by $[T]_\mathcal{B}$, we can obtain a matrix with respect to different bases by multiplying with the appropriate base change matrices in the same way as above:
\[
[T]_\mathcal{D} = [\text{id}]_\mathcal{C} \cdot [T]_\mathcal{B} \cdot [\text{id}]_\mathcal{D}
\]
In particular, if we set $\mathcal{C} = \mathcal{B}$ and $\mathcal{E} = \mathcal{D}$, we get that
\[
[T]_\mathcal{D} = [T]_\mathcal{D} = [\text{id}]_\mathcal{B} \cdot [T]_\mathcal{B} \cdot [\text{id}]_\mathcal{D} = [T]_\mathcal{B} \cdot Q^{-1} \cdot [T]_\mathcal{B} \cdot Q
\]
and
\[
[T]_\mathcal{D} = [T]_\mathcal{D} = [\text{id}]_\mathcal{B} \cdot [T]_\mathcal{B} \cdot [\text{id}]_\mathcal{D} = [T]_\mathcal{D} \cdot P^{-1} \cdot [T]_\mathcal{B} \cdot P
\]
where $Q = [\text{id}]_\mathcal{D}$.