Mathematics 567 **Homework** (due May 4) A. Hulpke

71*) Let F be a perfect field and $f \in F[x]$ irreducible of degree n.
Show that there is a field $F < K < F(\alpha)$ if and only if there exist polynomials $g, h \in F[x]$ of degree $< n$ such that $f(x)$ divides $g(h(x))$.

72) Let $F = \mathbb{Q}$ and $f(x) = x^4 - 24x^3 + 144x^2 - 288x + 144$. Then (you do not need to verify this) f is irreducible and has a root $\alpha = (2+\sqrt{2})(3+\sqrt{3})$, furthermore $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{(2)}, \sqrt{(3)})$ is the splitting field of f. Now consider the polynomial $g(x) = f(x^2)$ (which also is irreducible). Let $\beta = \sqrt{\alpha}$ be a root of $g(x)$ in its splitting field $K > \mathbb{Q}$.
a) Determine the Galois conjugates of α.
b) Consider $\beta_2 := \beta \cdot \frac{\sqrt{3}-1}{\sqrt{2}}$, $\beta_3 := \beta \cdot (\sqrt{2}-1)$ and $\beta_4 := \beta \cdot \frac{(1-\sqrt{2})(1-\sqrt{3})}{\sqrt{2}}$. By calculating β_i^2 for $i = 2, 3, 4$, show that the roots of $g(x)$ are $\pm\beta, \pm\beta_2, \pm\beta_3, \pm\beta_4$.
c) Conclude that $[K : \mathbb{Q}] = 8$. (Note that – problem 70 – this does not follow from $[\mathbb{Q}(\beta) : \mathbb{Q}(\alpha)] = 2$!)
d) Determine $\mathrm{Gal}(K/\mathbb{Q})$ and show that it is isomorphic to the quaternion group $\mathbb{Q}_8$.
e) Determine the subfields of K.

73) Construct a field extension of $\mathbb{Q}$ with Galois group $Z_3^2 \times Z_5$.

74*) a) Let $G \leq S_n$ containing an n-cycle, an $n-1$-cycle and a 2-cycle. Show that $G = S_n$.
b) Show that for every n there is a polynomial of degree n in $\mathbb{Q}[x]$ which is irreducible and has Galois group S_n.

75*) (GAP or another computer algebra system) Factorize the polynomial $-2 - 6x + 4x^2 + 15x^3 - 2x^4 - 8x^5 + x^7$ modulo 500 primes (excluding 2 and 809 which divide the discriminant) and tabulate the frequency of the occuring cycle shapes.
Based on this information, and the cycle type distribution for transitive subgroups of S_7 make a guess on the type of $\mathrm{Gal}(f)$.

76) Determine the subfields of the splitting field of $x^{35} - 1$.

Problems marked with *are bonus problems for extra credit.