Mathematics 566 **Homework (due Oct. 17th)** A. Hulpke

D&F;4.3: 5,13,19*,26

1) In this problem, we want to show that a subgroup of S_6 of index 6 must be isomorphic to S_5:
a) Using the fact that A_6 is simple (you don't need to prove this), show that the only normal subgroups of S_6 are $\langle 1\rangle$, A_6 and S_6.
b) Let $U \leq S_6$ be a subgroup of index 6. Let φ be the action of S_6 on the cosets of U. Using a), show that the kernel of φ must be the trivial subgroup and conclude that φ is an isomorphism $S_6 \to S_6$.
c) Show that the image of $\varphi(U)$ must be S_5, conclude that $U \cong S_5$.

2) In this problem, we will show that the group of rotations of an icosahedron is isomorphic to A_5.
a) Let H be the group of rotations and reflections of an icosahedron and $G \leq H$ the subgroup of rotations. (We have proven before that $|H| = 120$ and $[H : G] = 2$.) Let C be the set of pairs of *opposite corners* of the icosahedron. Let σ be the permutation representation of H, afforded by the action on C. Show that $H\sigma \leq S_6$.
b) Determine $\bigcap_{c\in C} \mathrm{Stab}_H(c)$ and show that σ is an isomorphism.
c) Conclude (using problem 1) that $H \cong S_5$ and that $G \cong A_5$.

3)* How many different possibilities (up to rotations only) are there to color the faces of an icosahedron with the colors red and green, coloring 10 faces red and 10 green?
Hint: From problem 2 we know that the acting group is isomorphic A_5. Use your knowledge about the classes of A_5 to show that a set of elements of certain orders must yield class representatives.

4) Let G be a group and $U \leq G$ with $[G : U] = 3$. Show that G posesses a normal subgroup of index 2 or 3.
Hint: Consider the action of G on the cosets of U. What are the possible image groups. Then use the isomorphism theorem.

5) Classify the transitive permutation representations of S_4: What is the degree? Is it faithful? (Can you give images for the generators $(1, 2, 3, 4), (1, 2)$ for each representation?)
You may use GAP for any calculations. Problems marked with a *are bonus problems.