Mathematics 566 **Homework (due Nov 7th)** A. Hulpke

D&F;7.1: 2,24*,28

D&F;7.3: 1,3,6,10,20*,26,34
Problems marked with a *are bonus problems.

Exercises for Group Theory

The following group theory problems are of a level of difficulty suitable for a final or the qualifyer. You don't have to hand solutions for these problems, but if you have problems with any, feel free to ask.

1) Show that every group of order 77 is cyclic.

2) Schow that $GL(3, \mathbb{Z}_5)$ has a normal subgroup of index 4.

3) Let p, q, r be prime numbers and $|G| = pqr$. Show that G is solvable.
(Hint: Distinguish the cases that p, q, r are different or that some of them are equal.)

4) A group of order 275 acts on a set of size 18. How many orbits of length 1 does it have at least?

5) Show that a group of order 108 or 351 is solvable.

6) a) Let G be a group and $G/Z(G)$ cyclic. Show that G is abelian.
b) Let $|G| = p^3$. Show that G is abelian, or $Z(G) = G'$.

7) Let $\pi = (1, 4, 5)(2, 3)(6, 7)$. Determine the orders $|C_{S_7}(\pi)|$ and $|C_{A_7}(\pi)|$.

8) Determine the conjugacy classes and normal subgroups of D_{10}.

9) Let $|G| = 1990$.
a) Show that G posesses a nontrivial normal subgroup.
b) Show that $G^{(3)}$ (the third iterated derived subgroup) is trivial.

10) Let $|G| = 6$. Show that G is abelian if and only if $\mathrm{Aut}(G)$ is abelian.

11) Let G be a finite group, p a prime and $N := \langle g \in G \mid p$ does not divide $|g|\rangle$. Show:
a) For every $\phi \in \mathrm{Aut}(G)$ we have that $\phi(N) = N$.
b) G/N is a p-group (i.e. it has size p^a for some a).

12) Let $G \leq S_n$. Show: If G contains an odd permutation, the there exists $N \lhd G$, $[G : N] = 2$.

13) Show: S_{n+2} has two (different) conjugacy classes of subgroups isomorphic S_n.

14) a) Show that G is abelian if and only if the map $\phi: G \to G$, $\phi(x) = x^{-1}$ is an automorphism of G.
b) Let $G \neq \{1\}$ be a finite group which only posesses the trivial automorphism. Determine the possible isomorphism types of G.

15) Let G be a group and $|G| = p_1^{n_1} \cdot p_2^{n_2} \cdot \cdots \cdot p_r^{n_r}$ with p_i (different) primes and $p_1 < p_2 < \cdots < p_r$. Show:
a) If $N \lhd G$ with $|N| = p_1$, then $N \leq Z(G)$.
b) If $U \leq G$, $[G:U] = p_1$, then $U \lhd G$. (Hint: Consider the action on the cosets of U and its image in S_{p_1}.)

16) What is the maximal order for an element in S_{17}?

17) Explain: The number of conjugacy classes in S_5 is the same as the number of abelian groups of order 32.

18) Let $G = GL(2,3)$ (the group of invertible 2×2 matrices over Z_3). Show (via the action on $\mathbb{Z}_3^2$) that $G/Z(G) \cong S_4$.

19) Let G be the direct product of its subgroups U, V. Show:
a) If $N \lhd U$ then $N \lhd G$.
b) $Z(G) = Z(U) \times Z(V)$.

20) Let $|G| = 300$. Show that G has a normal subgroup of size 5 or 25. (Consider the conjugation action on $\mathrm{Syl}_5(G)$.)

21) Let G be a finite group and $U \leq G$ the only maximal subgroup (i.e. for every $V \leq G$ either $V = G$ or $V \leq U$.) Show that G is cyclic of prime power order.

22) Let $M, N \lhd G$ with G/N and G/M solvable. Show that $G/(M \cap N)$ and $G/\langle M, N\rangle$ are solvable.

23) Determine the isomorphism types of abelian groups of size $1188 = 2^2 \cdot 3^3 \cdot 11$. Identify Z_{1188} and $Z_{33} \times Z_{36}$ in this list.

24) Determine the subgroup lattice of $GL(2,2)$.