Mathematics 501	**Final (100 points)**	Due 5/9/17, 4.10pm

Points (leave blank)

1	2	3	4	5	6	7	$\sum$

Name:

(clearly, please)

This exam is my own work. Sources (apart from class notes) are indicated. I have not given, received, or used any unauthorized assistance.

Signature

This final is due Tuesday, December 13th 2016 at 4.10pm (at my office, in my mailbox, or at the in-class final).

Notes

- Put your name on this cover sheet and sign it.
- All problems carry equal weight.
- A description of how you solved the problem, respectively justification of the steps taken, is a crucial part of every solution.
- You are permitted to use class notes and any publication (book, journal, web page). You are not permitted to consult third persons.
  Results which are quoted from a publication (apart from the course notes and your lecture notes) must be indicated.
- You may use a computer unless this renders the problem trivial.
- If you need extra sheets, staple them to this final. (You do not need to submit scrap paper.)
- I will put the graded finals with your course grades in your mailboxes in the math department if you have one. Otherwise you can pick up your exam in

1) Construct a BCH code over $\mathbb{F}_3$ of length 10 and designed distance 5.

2) You are given 12 coins, one of which is known to be either lighter or heavier than all the others; you are also given a beam balance. Devise a scheme of three weighings which will identify the odd coin and determine if it is light or heavy; the coins weighed at each step should not depend on the results of previous weighings. (What is the connection between this problem and error-correcting codes over $\mathbb{F}_3$?)

3) Suppose that $\mathcal{D}$ is a $t-(v,k,1)$ design. Show that the numbers

$$\binom{v}{t}\Big/\binom{k}{t}, \binom{v-1}{t-1}\Big/\binom{k-1}{t-1}, \ldots \binom{v-t+1}{1}\Big/\binom{k-t+1}{1}$$

must be integers. Conclude that if there is a $2-(v,3,1)$ design we must have that $v \equiv 1,3 \pmod 6$.

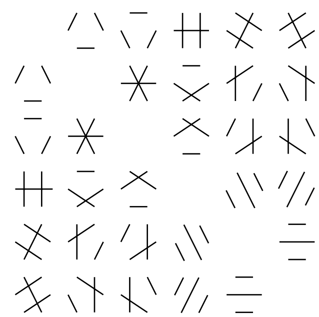
4) Let $G = C_3 \wr S_2 \leq S_9$, acting in product action on $3^2 = 9$ points.
a) Show that the cycle structures of the elements of G are (exactly) 1^9, $1^3 2^3$, 3^3 and $3 \cdot 6$.
b) Determine the cycle index for G in this action.
c) G acts (by row- and column-permutations and transposition) on 3×3 matrices whose entries are chosen from $0, 1, 2$. Determine the number of orbits.

5) For a projective plane of order n we form an incidence matrix $A \in \{0,1\}^{(n^2+n+1)\times(n^2+n+1)}$ with rows corresponding to lines and columns to points. Let $\mathcal{C}$ be the $\mathbb{F}_2$-code generated by the rows of this matrix.

a) Show that for odd n this code $\mathcal{C}$ consists of all the words of even weight.

b) Now consider $n \equiv 2 \pmod 4$. We form an extended code $\mathcal{C}'$ by adding to every code word one parity bit. Show that $\mathcal{C}' \subset \mathcal{C}'^{\perp}$.

6) Let $n = 6$. There are $\binom{6}{2} = 15$ pairs of points which we can consider as edges in the complete graph. A *factor* is a set of 3 disjoint (i.e. not shaing end points) edges. A *factorization* is a partition of the 15 edges into five factors.
a) Show that two disjoint factors are contained in a unique factorization.
b) Show that there are exactly six factorizations. (Compare the logo for the Combinatorics Seminar shown.)



c) Define an action of S_6 on factorizations. This gives a homomorphism $S_6 \to S_6$. Show that it is bijective.
d) Show that the automorphism constructed in c) is an outer automorphism. (Consider for example the image of $(1, 2)$ under the map).
(We thus conclude that S_6 has two nonequivalent transitive permutation actions on 6 points. One can show that this phenomenon is unique to the number 6, but this is not part of the task here.)

7) Show that $PGL(n+1, q)$ acts transitively and primitively on the lines in $PG(n, q)$.