21) The group of rotations of a dodecahedron, acting on its faces, has (you may take this as given) cycle index
\[ \frac{1}{60} \left( s_1^2 + 15s_2^2 + 24s_1^2 s_2 + 20s_3^4 \right) \]
In how many different ways, up to symmetry, can we color the faces of a dodecahedron so that 10 faces are white and 2 are red, green or blue?

22) Let $G$ be a permutation group on $\Omega$ and $H$ a permutation group on $\Delta$. We know that $G \times H$ acts intransitively on $\Omega \cup \Delta$. In that action the cycle structure of elements is simply the union of the cycle structures on both sets. Show that for these actions we have that
\[ Z(G \times H; s_1, s_2, \ldots) = Z(G; s_1, s_2, \ldots) \cdot Z(H; s_1, s_2, \ldots). \]

23) Let $\{ p_i \}$ be the power-sum symmetric functions and $\{ h_i \}$ the complete homogeneous symmetric functions. Show that
\[ h_n = Z(S_n; p_1, \ldots, p_n). \]

24) a) Using the fact that the polynomial $x^2 - x - 1$ is irreducible over $\mathbb{F}_3$, construct the addition and multiplication tables of a field with 9 elements.
b) Show that the polynomial $x^2 + x + 1$ has no root in $\mathbb{F}_3$ but has a root in the field constructed under a).

25) Show that for $0 \leq k \leq n$ and $q$ a prime power we have that
\[ \begin{bmatrix} n+1 \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n \\ k \end{bmatrix}_q \]
and
\[ \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q \]