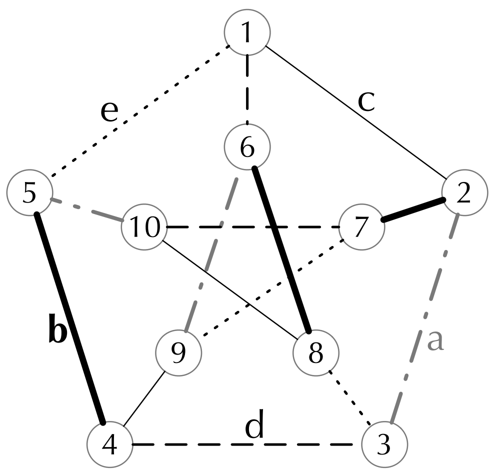
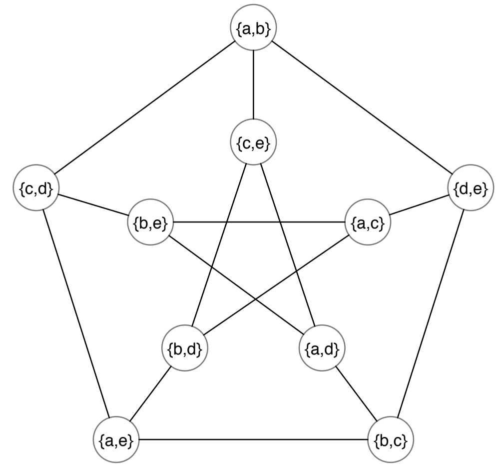
6) In this exercise we want to show that the automorphism group A of the Petersen graph P is isomorphic to S_5.
a) The figure below, left depicts five subsets of edges, labelled by $a, \ldots, e$. Show that any automorphism of P must permute these sets. (**Hint:** Find a characterization of the sets that does not rely on the picture, e.g. how you can find the two other edges if one edge is given.)
Thus we have a homomorphism $\phi: A \to S_5$.
b) The right figure depicts a characterization of P as vertices labelled with the $\binom{5}{2}$ two-element subsets of $\{a, b, c, d, e\}$, sets being connected if and only they are disjoint. Show that this yields a homomorphism $\psi: S_5 \to A$.
c) Show that ϕ and ψ are mutually inverse, that is that $A \cong S_5$.
We note that the characterization from b) provides a nice way of writing down automorphisms of P.





7) Let G be a group with generating set A. Show that G acts on the (vertices of) the Cayley graph $\mathrm{Cay}_A(G)$ by $g: v \mapsto g^{-1}v$, and that this action is compatible with arrows and their labelling.

8) Consider $G = S_3 \times S_3$ as an intransitive permutation group on 6 points and let $H = G \cap A_6$ the subgroup of even permutations. What is $|H|$? Show that H has two orbits of size 3, but is not a direct product of the images of its actions on these two orbits.

9) Let G be transitive on Ω and $N \lhd G$. Show that the orbits of N on Ω form a block system for G. Conclude that a nontrivial normal subgroup of a primitive group must act transitively.

10) (Continuation of problem 5). Let R be the group of rotational symmetries of an icosahedron and $S \geq R$ the group of rotational and reflective symmetries. (We know already that $|R| = 60$ and $|S| = 120$.) The shading of faces in the illustration indicates five possible ways to embed a tetrahedron (with corners on the middle points of faces) into an icosahedron.

a) Show that R acts transitively on these five embedded tetrahedra (the corresponding faces thus form blocks in a block system), and that there thus is a homomorphism $\varphi\colon R \to S_5$.

b) Show that the image $\varphi(R)$ consists only of even permutations and conclude that $R \cong A_5$.

c) Show that the permutation

$$s = (1,20)(2,12)(3,17)(4,16)(5,19)(6,14)(7,13)(8,18)(9,15)(10,11)$$

represents a reflection of the icosahedron, and thus $s \in S \setminus R$.

d) Show that s commutes with the rotational generator p_1, p_2, and conclude that $S \cong 2 \times A_5$.