**Mathematics 501** **Final (70 points)** Due 12/13/16, 4.10pm

| Points (leave blank) | | | | | | | |
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| 1 | 2 | 3 | 4 | 5 | 6 | 7 | $\sum$ |
| | | | | | | | |

Name:

(clearly, please)

This exam is my own work. Sources (apart from class notes) are indicated. I have not given, received, or used any unauthorized assistance.

Signature

This final is due Tuesday, December 13th 2016 at 4.10pm (at my office, in my mailbox, or at the in-class final).

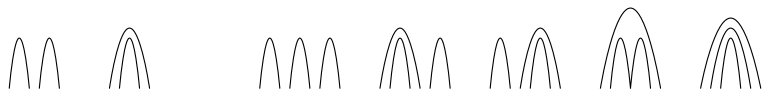
# Notes

- Put your name on this cover sheet and sign it.
- All problems carry equal weight.
- A description of how you solved the problem, respectively justification of the steps taken, is a crucial part of every solution.
- You are permitted to use class notes and any publication (book, journal, web page). You are not permitted to consult third persons.
  Results which are quoted from a publication (apart from the course notes and your lecture notes) must be indicated.
- You may use a computer unless this renders the problem trivial.
- If you need extra sheets, staple them to this final. (You do not need to submit scrap paper.)
- I will put the graded finals with your course grades in your mailboxes in the math department if you have one. Otherwise you can pick up your exam in

**1)** This is a variant of the problem
`http://fivethirtyeight.com/features/can-you-survive-this-deadly-board-game`
with changed coefficients to make the calculation easier. I am looking for an exact result, not a simulation or a numerical approximation.
We take an ordinary die and relabel the faces $4, 5, 6$ with $2$. We now consider the following process: On a track with fields labelled from 0 to 1000, we place a counter on 0. We then roll our modified die repeatedly, and have the counter step forwards according to the result (assume fields $1001 \ldots 1004$ to deal with overstepping at the end).
a) Determine an expression that for any field $n$ gives the probability $p_n$ that the counter would hit that field during the process.
(**Hint:** Consider the power series $F(t) = \sum_{n=1}^{\infty} p_n t^i$ and the polynomial $f = \frac{1}{6}(t + 4t^2 + t^3)$. Show that $F(t) = \sum_{i=1}^{\infty} f^i$. Using the geometric series, determine an expression for $F(t)$ as a rational function.)
b) What number in the interval $[1, \ldots, 1000]$ has the highest chance to be hit?

**2)** Consider the coefficients $N_{\lambda\mu}$ in the expression $h_\lambda = \sum_{\mu \vdash n} N_{\lambda\mu} m_\mu$. Show that $N_{\lambda\mu}$ counts the number of solutions to the following counting problem: We have $n$ balls, of which $\lambda_i$ are labelled with number $i$. In how many different ways (assuming balls with the same number to be indistinguishable) can we place the balls in boxes $1, 2, \ldots$ such that box $i$ contains exactly $\mu_i$ balls.

**3)** We construct a notebook (with $4n$ pages) by folding $n$ blank sheets in half and binding them together. Determine the number of ways this can be done for $n$ sheets. The picture below shows the possibilities for $n = 2$ and $n = 3$:



**4)** For $r \leq n$, an $r \times n$ array $A = (a_{i,j})$ is a *latin rectangle*, if every row contains the numbers 1 to $n$ in some order, and each column contains no number repeatedly.
Show that any latin rectangle can be completed to a latin square by adding further rows.

**5)** The careless users of a library (assume it to be one long bookshelf) have placed some books back in the wrong places (i.e. a permutation $\pi$ of the books is given), and the task exists to bring it back in order. This is done by repeated application of the operation of taking one book out and placing it back in another position (shifting books to make space). Show that the library can be ordered by $k$ such remove/insert operations if and only if $\pi$ has an increasing subsequence of length $n - k$.

**6)** Show the following theorem (which is a dual version of Dilworth's theorem): Let $P$ be a partially ordered set. If $P$ possesses no chain of $m+1$ elements, then $P$ is the union of m antichains.

**7)** Consider the permutation (as sequence) $[2, 5, 6, 7, 3, 4, 1]$. In the RSK correspondence, it corresponds to two tableaux $P, Q$. Determine the permutation that corresponds to $P^T, Q^T$.