Mathematics 467 **Homework (due Apr. 3)** A. Hulpke

42) Let α be a root of $x^3 - x^2 + 5$ and let $K = \mathbb{Q}(\alpha)$.
a) Show that $1, \alpha, \alpha^2$ is a $\mathbb{Q}$-basis of K.
b) Represent the following elements of K as $\mathbb{Q}$-linear combinations of the basis elements in a): (You may use a computer algebra system for polynomial arithmetic.)

$$2\alpha + 5, \quad \alpha^4, \quad 1/\alpha, \quad (3+4\alpha)/(2\alpha^2+1)$$

c) Determine the minimal polynomial of $2 + 3\alpha$.

43) Let $F \leq E$ be a field extension of degree n.
a) For any $\alpha \in E$ prove that multiplication by α is an F-linear transformation of E.
b) Show that there is an injective ring homomorphism $\varphi E \to F^{n\times n}$.
c) Show that for $\alpha \in E$ the minimal polynomial of α over F (in the sense of field extensions) equals the minimal polynomial of the matrix α^φ.
d) Determine the minimal polynomials over $\mathbb{Q}$ for $\sqrt[3]{2}$ and $1 + \sqrt[3]{2} + \sqrt[3]{4}$.

44) Let $E = \mathbb{Q}(\sqrt[3]{5}, \zeta)$ where $\zeta = e^{\frac{2\pi i}{3}}$. Determine a $\mathbb{Q}$-basis of E. What is $[E : \mathbb{Q}]$?

45) Determine the splitting field of $x^4 + 2$ over $\mathbb{Q}$.

46) Let $K = \mathbb{Q}(\sqrt{2})$ and let

$$\varphi\colon K \to K, \quad a + b\sqrt{2} \mapsto a - b\sqrt{2}$$

Show that φ is a field automorphism (i.e. a bijective field homomorphism) of K.

Problems marked with a *are bonus problems for extra credit.

Note: More details on Galois theory can be found for example in the textbook *Galois Theory* by S. Weintraub, which is available for free electronically from the university library: On the campus network, go to the link: `http://www.springer.com/math/algebra/book/978-0-387-87574-3` respectively `http://tinyurl.com/dzq8eb` (Follow the "Online Version" link).