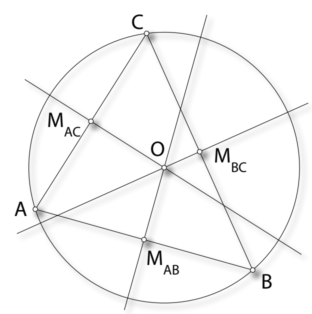
36) Let $I = \langle xy^3 - x^2, x^3y^2 - y\rangle \lhd \mathbb{Q}[x,y] = R$. a) Compute a Gröbner basis for I with respect to GRLEX ordering. Determine a cononical form of coset representatives for R/I. What is the dimension d of R/I as a $\mathbb{Q}$ vector space?
b) The maps $\alpha\colon R/I \to R/I, I+p \mapsto I+p\cdot x$ and $\beta\colon R/I \to R/I, I+p \mapsto I+p\cdot y$ are $\mathbb{Q}$-vector-space homomorphisms of R/I. (Persuade yourself that they are, but you do not need to show this in the solution.) They therefore can be represented by matrices. Determine matrices for M_α and M_β with respect to the basis for R/I determined in a).
c) Show that the map $\varphi\colon R \to \mathbb{Q}^{d\times d}, f(x,y) \mapsto f(M_\alpha, M_\beta)$ is a ring homomorphism with kernel I. (In other words: we can compute in R/I by computing with these matrices instead.)

37) Let $R = \mathbb{Q}[x,y,z]$ and $I = \langle x^2 + yz - 2, y^2 + xz - 3, xy + z^2 - 5\rangle \lhd R$. Show that $I+x$ is a unit in R/I and determine $(I+x)^{-1}$.

38) Suppose that A, B, C, D are points in the plane. Write polynomial equations in the coordinates of these points that describe:
a) C is collinear with $\overline{AB}$.
b) $\overline{BD}$ bisects $\angle ABC$.



39*) A theorem of EUCLID states, that for a triangle ABC the lines bisecting the sides perpendicularly intersect in the center of the outer circle (the circle through the vertices) of the triangle. Prove this theorem using coordinates and polynomials.
Hint: For reasons of symmetry, it is sufficient to assume that *two* of the perpendicular lines intersect in this point, or that the center lies on each perpendicular line. You may also assume by scaling and rotating that $A = (0,0)$ and $B = (0,1)$.

40) Let F be a field and $R = F[x_1, \cdots, x_n]$ and let $I_1 = \langle f_1, \ldots, f_k\rangle \lhd R$ and $I_2 = \langle h_1, \ldots, h_r\rangle \lhd R$ be two ideals.
a) Let $S = F[x_1, \cdots, x_n, t]$ (considering $R \subset S$; i.e. we introduce a further auxillary variable t) and set

$$J = \langle t\cdot f_1, \ldots, t\cdot f_k, (1-t)\cdot h_1, \ldots, (1-t)\cdot h_r\rangle \lhd S.$$

Let $g \in J \cap R$ (i.e. g does not involve t). Show that $g(x_1, \ldots, x_n, 0) = g(x_1, \ldots, x_n, 1)$ and conclude that $g \in I_1 \cap I_2$.
b) Vice versa, suppose that $g \in I_1 \cap I_2$. We can thus write (for suitable polynomials $\{c_i\}, \{d_j\} \subset R$:

$$g = \sum_i c_i f_i = \sum_j d_j g_j$$

Using $g = tg + (1-t)g$, show that $g \in J$ (and thus, with a), $I_1 \cap I_2 = J \cap R$).

41) (Example for 40)) By problem 35, if G is a LEX Gröbner basis (with $y > x_i$) for J, then $G \cap R$ (the polynomials not involving y) is a basis for $J \cap R$.
a) Let $f = x^3z^2 + x^2yz^2 - xy^2z^2 - y^3z^2 + x^4 + x^3y - x^2y^2 - xy^3$ and
$g = x^2z^4 - y^2z^4 + 2x^3z^2 - 2xy^2z^2 + x^4 - x^2y^2$.
Compute $\langle f\rangle \cap \langle g\rangle$.
b) Compute $\gcd(f,g)$. (Hint: Show that $\langle f\rangle \cap \langle g\rangle = \langle \operatorname{lcm}(f,g)\rangle$.)

Problems marked with a *are bonus problems for extra credit.