31) A basis $\{\underline{\mathbf{x}}^{\alpha_1},\ldots,\underline{\mathbf{x}}^{\alpha_s}\}$ for a monomial ideal I is said to be *minimal* if no $\underline{\mathbf{x}}^{\alpha_i}$ in the basis divides another $\underline{\mathbf{x}}^{\alpha_j}$ for $i \neq j$.
a) Prove that every monomial ideal has a minimal basis.
b) Show that every monomial ideal has a *unique* minimal basis (up to arrangement).

32) Let $I = \langle x^2y-1, xy^2-x\rangle \lhd \mathbb{Q}[x,y]$. We are using GRLEX ordering. Compute (without using a `GroebnerBasis` routine!) a Gröbner basis for I. You may use the computer for polynomial arithmetic, as well as to perform the division algorithm. (I.e. if you just state what the S-polynoms and their reductions are, this is fine.)
In GAP, if `order` is the ordering (i.e. `order:=MonomialGrlexOrdering();`), you can divide for example the polynomial $p := x^4y^4$ by $I = \langle f,g\rangle$ with the command

```
gap> r:=PolynomialDivisionAlgorithm(p,[f,g],order);
[ y^2, [ x^2*y^3+y^2, 0 ] ]
```

The first entry (`r[1]`= y^2) is the remainder, the second entry the list of quotients. For computing the S-polynomial, the following command computes the leading monomial:

```
gap> LeadingMonomialOfPolynomial(f,order);
x^2*y
```

33*) (This problem is to illustrate a reason for having different monomial orderings. You don't need to submit the full results, but just a brief description of what happens.)
Let $I = \langle x^5+y^4+z^3-1, x^3+y^2+z^2-1\rangle$. Compute (e.g. with GAP) a (reduced) Gröbner basis for I with respect to the LEX and GRLEX orderings. Compare the number of polynomials and their degrees in these bases.
Repeat the calculations for $I = \langle x^5+y^4+z^3-1, x^3+y^3+z^2-1\rangle$ (only one exponent changed!)

34) Consider the equations for the robot arm in problem 18. Considering a and b as constants, determine values for the angles (respectively: For sine and cosine of the angles) that will place the arm at position (a,b).

35) Suppose that $R = \mathbb{Q}[x_1,\ldots,x_n]$ and $S = \mathbb{Q}[x_1,\ldots,x_n,t]$ (i.e. one extra variable). We consider $R \subset S$.
a) We will use a LEX ordering with $t > x_1 > x_2 > \cdots > x_n$. Let $f \in S$. Show that if $LT(f)\mathbb{R}$, then $f \in R$.
b) Let $I \lhd S$. Show that $J := I \cap R \lhd R$.
c) Show that $\langle LT(I)\rangle \cap R = \langle LT(I\cap R)\rangle$.
d) Suppose that G is a Gröbner basis for I. Show that $G \cap R$ is a Gr"obner basis for $I \cap R$. (This is a more formal justification for the variable elimination condition when using LEX-ordering.)

Problems marked with a *are bonus problems for extra credit.