24) Rewrite the following polynomial, ordering its terms according to the LEX and GRLEX ordering and give $LM(f)$, $LT(f)$ and $\mathrm{mdeg}(f)$ in each case.

$$f(x,y,z) = 2x^2y^8 - 3x^5yz^4 + xyz^3 - xy^4.$$

25) Let $B = (x^2y - z, xy - 1)$ and $f = x^3 - x^2y - x^2z + x$. a) Compute the remainder of dividing f by B for both the LEX and the GRLEX ordering.
b) Repeat part a) with the order of the pair B reversed.

26) Let $I = \langle x^3y^6, x^5y^4, x^6, x^4y^7\rangle$. Use the method of Dickson's Lemma to find an ideal basis for I.

27) In GRLEX ordering with $x > y > z$, is $\{x^4y^2 - z^5, x^3y^3 - 1, x^2y^4 - 2z\}$ a Gröbner basis?

28) Compute the S-polynomial for $x^4y - z^2$ and $3xz^2 - y$ with respect to the LEX ordering with $x > y > z$ and for $z > y > x$.

29) Let $I \lhd k[x_1, \ldots, x_n]$ be a principal ideal (i.e. there is $f \in I$ such that $I = \langle f\rangle$). Show that any finite subset of I containing a generator of I is a Gröbner basis for I.

30) Is $x^3z - 2y^2$ in the ideal $\langle xz - y, xy + 2z^2, y - z\rangle$?

Gröbner bases in GAP GAP has functions to compute Groebner bases for several orderings (which you are welcome to use in the homework).
You define an ordering by a call to

```
gap> myord:=MonomialLexOrdering();
MonomialLexOrdering()
```

respectively `MonomialGrlexOrdering()` . Don't forget the parentheses – these are functions which return an ordering. By default orderings are defined to have the variables defined first being larger than variables defined later. (This typically ensures that $x > y > z$.) It is possible, however to specify variables in decreasing order as arguments to the function call to define an ordering based on nonstandard variable ordering. For example

```
gap> myord2:=MonomialLexOrdering(z,y,x);
MonomialLexOrdering([ z, y, x ])
```

The function `LeadingTermOfPolynomial` returns the leading term (depending on the ordering):

```
gap> f:=2*x+3*y+4*z+5*x^2-6*z^2+7*y^3;
7*y^3+5*x^2-6*z^2+2*x+3*y+4*z
gap> LeadingTermOfPolynomial(f,myord);
5*x^2
gap> LeadingTermOfPolynomial(f,myord2);
-6*z^2
```

`PolynomialReduction` can be used to determine remainders. The first entry is the remainder, the second the coefficients with respect to the list of basis elements. (Note that the division algorithm works slightly different than the one in the book, thus if G is not a Gröbner basis you might get different remainders.) `PolynomialReducedRemainder` returns the remainder only.

```
gap> B:=[x*y-y^2,y^2-x];
[ x*y-y^2, y^2-x ]
gap> PolynomialReduction(x^5*y,B,myord);
[ y^6, [ x^4+x^3*y+x^2*y^2+x*y^3+y^4, 0 ] ]
gap> PolynomialReducedRemainder(x^5*y,B,myord);
y^6
```

The commands `GroebnerBasis` and `ReducedGroebnerBasis` compute Gröbner bases for a given generating set and ordering.

```
gap> GroebnerBasis(B,myord);
[ x*y-y^2, y^2-x, y^3-y^2 ]
gap> G:=ReducedGroebnerBasis(B,myord);
[ y^3-y^2, -y^2+x ]
```

By setting an appropriate `InfoLevel` higher, it is possible to get some information about the computation:

```
gap> SetInfoLevel(InfoGroebner,3);
gap> G:=ReducedGroebnerBasis(B,myord);
#I  Spol(2,1)=y^3-y^2
#I  reduces to y^3-y^2
#I  |bas|=3, 4 pairs left
#I  Spol(3,1)=y^4-x*y^2
#I  Pair (3,2) avoided
#I  Spol(3,3)=0
[ y^3-y^2, -y^2+x ]
```