Mathematics 467 **Homework (due Feb. 27)** A. Hulpke

19) a) Let $f(x) \in \mathbb{Z}[x]$ be irreducible. Show that there are only finitely many primes p, such that the reduction of f modulo p has multiple roots. (Hint: Discriminant)
b) Determine all such primes for the polynomial $x^7 + 15x^6 + 12$.

20) Find all rational solution to the following system of equations:

$$\{x^2y - 3xy^2 + x^2 - 3xy = 0, x^3y + x^3 - 4y^2 - 3y + 1 = 0\}$$

You may use a computer algebra system to calculate resultants or factor polynomials, for example in GAP the functions `Resultant` and `Factors` might be helpful. (See the online help for details.)

21) Consider the parametric curve

$$x = \frac{1-t^2}{1+t^2}, \qquad y = \frac{2t}{1+t^2}.$$

Describe this curve by polynomials in x,y, and t. By eliminating t, determine a polynomial in x and y describing the curve and use this result to identify the curve.

22) Consider the curve, described in polar coordinates by the equation $r = 1 + \cos(2\theta)$. We want to describe this curve by an equation in x and y:
a) Write equations for the x- and y-coordinates of points on this curve parameterized by θ.
b) Take the equations of a) and write them as polynomials in the new variables $\sin\theta = z$ and $\cos\theta = w$.
c) Using the fact that $z^2 + w^2 = 1$, determine a single polynomial in x and y whose zeroes are the given curve. Does this polynomial have factors (GAP command `Factors`)?

23) Let $f\colon \mathbb{R}^n \to \mathbb{R}$. A point $x \in \mathbb{R}^n$ is called a *critical point* of f, if $\frac{\partial f}{\partial x_i}(x) = 0$. (Cf. Calculus 3.) Determine all critical points of the function

$$f(x,y) = (x^2 + y^2)^3 - 4 * x^2 * y^2$$

Note: this curve is the "four-leaved flower" $r = \sin(2\theta)$ in polar coordinates. Is there a geometric interpretation of the critical points?