14∗) (Partial Fractions)
a) Let $f, g, h \in \mathbb{Q}[x]$ with $a = \deg(g)$, $b = \deg(h)$ and $a + b > \deg(f)$. Show that if $\gcd(g, h) = 1$, then there are $r, s \in \mathbb{Q}[x]$ such that

$$f(x) = r(x)g(x) + s(x)h(x)$$

and $\deg r < b$ and $\deg h < a$. (Hint: If $1 = c \cdot g + d \cdot h$, divide $f \cdot c$ by $h$ with remainder.)
b) Let $r, p \in \mathbb{Q}[x]$. Show that we can write

$$r(x) = s_1(x)p^1(x) + s_2(x)p^{l-1}(x) + \cdots + s_{l-1}(x)p^2(x) + s_l(x)p(x) + s_{l+1}(x)$$

with $s_i \in \mathbb{Q}[x]$ and $\deg s_i(x) < \deg p(x)$.

c) Let $f, g_1, g_2, \ldots, g_k \in \mathbb{Q}[x]$, $\gcd(g_i, g_j) = 1$ for $i \neq j$ and $1 \leq e_i \in \mathbb{Z}$ such that $\deg f < \sum_i e_i \cdot \deg g_i$. Show that we can write (partial fraction decomposition, as done in calculus):

$$\frac{f(x)}{g_1^{e_1}(x) \cdot g_2^{e_2}(x) \cdots g_k^{e_k}(x)} = \frac{s_{1,1}(x)}{g_1(x)} + \frac{s_{1,2}(x)}{g_2(x)} + \cdots + \frac{s_{1,e_1}(x)}{g_1^{e_1}(x)} + \frac{s_{2,1}(x)}{g_2(x)} + \cdots + \frac{s_{2,e_2}(x)}{g_2^{e_2}(x)} + \cdots + \frac{s_{k,1}(x)}{g_k(x)} + \cdots + \frac{s_{k,e_k}(x)}{g_k^{e_k}(x)}$$

with $\deg s_{i,j} < \deg g_i$.

15) Let $f(x), g(x) \in \mathbb{Q}[x]$. Suppose that $\alpha, \beta \in \mathbb{C}$ such that $f(\alpha) = 0$, $g(\beta) = 0$.

a) Show that $\text{Res}(f(x - y), g(y), y)$ has a root $\alpha + \beta$. (Note: Similar expressions exist for $\alpha - \beta$, $\alpha \cdot \beta$ and $\alpha/\beta$.)
b) Construct a polynomial $f \in \mathbb{Q}[x]$ such that $f(\sqrt{5} + \sqrt{2}) = 0$.

16) Let $F$ be a field. For a polynomial

$$f = \sum_{i=0}^{n} a_ix^i \in F[x]$$

we define the derivative $Df$ as

$$Df = \sum_{i=1}^{n} i \cdot a_ix^{i-1}$$

(with $i$ being the corresponding sum of 1s.)

a) Show that for $f, g \in F[x]$ we have that $D(fg) = (Df)g + f(Dg)$.
b) Suppose that $f = a_n \prod(x - \alpha_i)$ in $F$. Show that $f$ has repeated roots if and only if $f$ and $Df$ have a nonconstant gcd.
17) The **discriminant** of a polynomial \( f \in F[x] \) of degree \( m \) with leading coefficient \( a \) is defined as

\[
\text{disc}(f) = (-1)^{\frac{m(m-1)}{2}} \text{Res}(f, Df)/a
\]

with \( Df \) defined as in problem 16.

a) Show that if \( f = \prod_i (x - \alpha_i) \), we have that \( \text{disc}(f) = \prod_{i < j} (\alpha_i - \alpha_j)^2 \).

b) Show that \( \text{disc}(f) = 0 \) if and only if \( f \) has multiple roots (possibly in a larger field than \( F \)).

18) Consider a (2-dimensional) robot arm as depicted.

We want to find out the angles \( \theta_1, \theta_2 \) to which the joints have to be set to move the hand to coordinates \((a, b)\). For simplification, assume the two arms have length \( l_1 = 3, l_2 = 4 \). To avoid using trigonometric functions, set \( s_i = \sin(\theta_i), \ c_i = \cos(\theta_i) \) \((i = 1, 2)\). Then \( s_i^2 + c_i^2 = 1 \).

Write down equations that determine \( a, b \) in terms of the variables \( c_1, s_1, c_2, s_2 \). (You will have to use the formulas for \( \sin(\alpha + \beta) \) and \( \cos(\alpha + \beta) \).)

Problems marked with a * are bonus problems for extra credit.