14*) (Partial Fractions)
a) Let $f, g, h \in \mathbb{Q}[x]$ with $a = \deg(g)$, $b = \deg(h)$ and $a + b > \deg(f)$. Show that if $\gcd(g, h) = 1$, then there are $r, s \in \mathbb{Q}[x]$ such that

$$f(x) = r(x)g(x) + s(x)h(x)$$

and $\deg r < b$ and $\deg h < a$. (**Hint:** If $1 = c \cdot g + d \cdot h$, divide $f \cdot c$ by h with remainder.)
b) Let $r, p \in \mathbb{Q}[x]$. Show that we can write

$$r(x) = s_1(x)p^l(x) + s_2(x)p^{l-1}(x) + \cdots + s_{l-1}(x)p^2(x) + s_l(x)p(x) + s_{l+1}(x)$$

with $s_i \in \mathbb{Q}[x]$ and $\deg s_i(x) < \deg p(x)$.
c) Let $f, g_1, g_2, \ldots, g_k \in \mathbb{Q}[x]$, $\gcd(g_i, g_j) = 1$ for $i \neq j$ and $1 \leq e_i \in \mathbb{Z}$ such that $\deg f < \sum_i e_i \cdot \deg g_i$. Show that we can write (partial fraction decomposition, as done in calculus):

$$\begin{aligned}\frac{f(x)}{g_1^{e_1}(x) \cdot g_2^{e_2}(x) \cdots g_k^{e_k}(x)} = \quad & \frac{s_{1,1}(x)}{g_1(x)} + \frac{s_{1,2}(x)}{g_1^2(x)} + \cdots + \frac{s_{1,e_1}(x)}{g_1^{e_1}(x)} \\ + \quad & \frac{s_{2,1}(x)}{g_2(x)} + \cdots \frac{s_{2,e_2}(x)}{g_2^{e_2}(x)} + \cdots \\ + \quad & \frac{s_{k,1}(x)}{g_k(x)} + \cdots \frac{s_{k,e_k}(x)}{g_k^{e_k}(x)}\end{aligned}$$

with $\deg s_{i,j} < \deg g_i$.

15) Let $f(x), g(x) \in \mathbb{Q}[x]$. Suppose that $\alpha, \beta \in \mathbb{C}$ such that $f(\alpha) = 0$, $g(\beta) = 0$.
a) Show that $\mathrm{Res}(f(x - y), g(y), y)$ has a root $\alpha + \beta$. (Note: Similar expressions exist for $\alpha - \beta$, $\alpha \cdot \beta$ and α/β.)
b) Construct a polynomial $f \in \mathbb{Q}[x]$ such that $f(\sqrt{5} + \sqrt[3]{2}) = 0$.

16) Let F be a field. For a polynomial

$$f = \sum_{i=0}^{n} a_i x^i \in F[x]$$

we define the *derivative* Df as

$$Df = \sum_{i=1}^{n} i \cdot a_i x^{i-1}$$

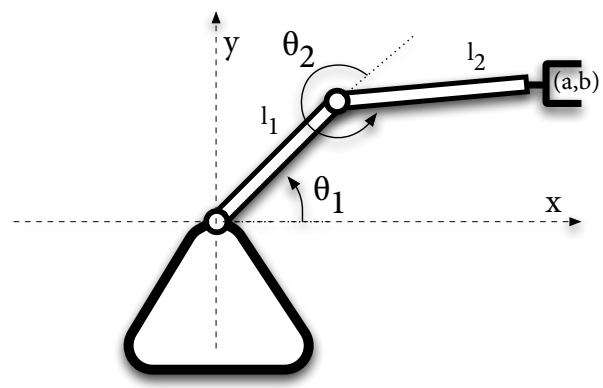
(with i being the the corresponding sum of 1s.)
a) Show that for $f, g \in F[x]$ we have that $D(fg) = (Df)g + f(Dg)$.
b) Suppose that $f = a_n \prod(x - \alpha_i)$ in F. Show that f has repeated roots if and only if f and Df have a nonconstant gcd.

17) The *discriminant* of a polynomial $f \in F[x]$ of degree m with leading coefficient a is defined as

$$\operatorname{disc}(f) = (-1)^{\frac{m(m-1)}{2}} \operatorname{Res}(f, Df)/a$$

with Df defined as in problem 16.
a) Show that if $f = \prod_i (x = \alpha_i)$, we have that $\operatorname{disc}(f) = \prod_{i<j} (\alpha_i - \alpha_j)^2$.
b) Show that $\operatorname{disc}(f) = 0$ if and only if f has multiple roots (possibly in a larger field than F).

18) Consider a (2-dimensional) robot arm as depicted.
We want to find out the angles θ_1, θ_2 to which the joints have to be set to move the hand to coordinates (a, b). For simplification, assume the two arms have length $l_1 = 3$, $l_2 = 4$. To avoid using trigonometric functions, set $s_i = \sin(\theta_i)$, $c_i = \cos(\theta_i)$ $(i = 1, 2)$. Then $s_i^2 + c_i^2 = 1$.
Write down equations that determine a, b in terms of the variables c_1, s_1, c_2, s_2. (You will have to use the formulas for $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$.)



Problems marked with a *are bonus problems for extra credit.