Mathematics 467 **Homework (due Feb. 13)** A. Hulpke

7) a) Show that the symmetric group S_3 is not free on the generators $(1,2),(1,2,3)$.
b) Let G be a free group on the free generators x, y. Show that G is not free on the three generators x,y,xy.

8) Show that for coprime numbers a, b we have that $C_{ab} \cong C_a \times C_b$.

9) a) Let

$$M = \begin{pmatrix} 4 & 6 & -2 \\ -6 & -15 & 6 \\ 4 & 24 & -11 \end{pmatrix}.$$

By row- and column-operations over $\mathbb{Z}$, convert M into a diagonal matrix.
b) Let F be a free $\mathbb{Z}$-module with free generators x_1, x_2, x_3 and let

$$S = \langle 4x_1 + 6x_2 - 2x_3, -6x_1 - 15x_2 + 6x_3, 4x_1 + 24x_2 - 11x_3 \rangle \leq F.$$

Determine the structure of F/S.

10) Let

$$\begin{aligned} G \; = \; & \langle a := (2,5,10,9)(3,12,7,11), b := (1,4)(2,3,10,7)(5,12,9,11)(6,8), \\ & c := (1,8,4,6)(2,12)(3,9)(5,7)(10,11) \rangle . \end{aligned}$$

This is an abelian group of order 32. Determine the orders of all possible products of the generators and write these as rows in a matrix. (E.g. $|a| = 4$ would be $(4,0,0)$, $|ab| = 4$ would be $(4,4,0)$ etc.) Use this matrix to determine the structure of G as a direct product of cyclic groups and determine from the transforming matrices elements of G that correspond to this structure. You may use GAP for element arithmetic and for computing the SNF.

11) Determine the abelian groups of order $48 = 2^4 \cdot 3$ up to isomorphism.

12*) Let G be the additive group of the rationals. Show that G is not finitely generated.

13) Let $p(n)$ be the number of partitions of order n.
a) Show that $p(n) \leq \sum_{i=1}^{n} p(n-1)$
b) Deduce that for any prime p the number (up to isomorphism) of abelian groups of order p^m is at most 2^m.
c) Show that there are at most n abelian groups of order n.
Problems marked with a *are bonus problems for extra credit.