52) (4 points) Let $p = x^4 - 2$ and E be the splitting field of p over $\mathbb{Q}$.
a) Show that $E = \mathbb{Q}(\sqrt[4]{2}, i)$ and express p as a product of linear factors over E.
b) Determine $[E : \mathbb{Q}]$.
c) Determine all field automorphisms of E over $\mathbb{Q}$.
d) For each automorphism determined in c), write the corresponding permutation of the roots of p (cf. problem 50).
e) Let G be the group generated by the permutations in d). (By the argument in problem 50, it is isomorphic to $\mathrm{Aut}(E/\mathbb{Q})$.) Determine all subgroups of G, for example in GAP, you may use the command:

```
sub:=Union(List(ConjugacyClassesSubgroups(G),Elements));
```

f) Draw the lattice (i.e. inclusion relation) amongst the subgroups. (You can use for example the GAP commands `IsSubset(sub[5],sub[4]);` to test inclusion)
g) For each subgroup $S \leq G$, determine the corresponding subgroup $T \leq \mathrm{Aut}(E/\mathbb{Q})$ (use the translation from d)!) and $\mathbb{Q} \leq (Fix)(T) \leq E$.
h) Sketch the inclusion relation amongst these subfields. Compare with f).

53) Determine the irreducible polynomials of degree dividing 4 over $\mathbb{F}_2$ and show that their product is $x^{16} - x$.
(**Hint:** You can verify the result in GAP:

```
x:=X(GF(2),"x");
Factors(x^16-x);
```

54) a) For positive integers d, n prove that d divides n if and only if $x^d - 1$ divides $x^n - 1$ (use $n = qd + r$).
b) Setting $x = p$ for a prime p, show that d divides n if and only if $p^d - 1$ divides $p^n - 1$.
c) Show (this is using parts a) and b)), that $x^{p^d} - x$ divides $x^{p^n} - x$ if and only if d divides n.
We have seen in Problem 48, that for every prime power p^n there is a field with p^n elements, namely the splitting field of $x^{p^n} - x$ over $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. We call this field $\mathbb{F}_{p^n}$.
d) Show that $\mathbb{F}_{p^d} \subset \mathbb{F}_{p^n}$ if and only if d divides n.
(Thus, for examplem $\mathbb{F}_2 \leq \mathbb{F}_4 \leq \mathbb{F}_{16}$, but $\mathbb{F}_4 \not\leq \mathbb{F}_8 \not\leq \mathbb{F}_{16}$.)

Problems marked with a *are bonus problems for extra credit.