Mathematics 467 **Homework (due Feb. 6)** A. Hulpke

1) a) Let

$$A := \begin{pmatrix} 60 & -60 & 48 & -232 \\ -9 & 9 & -6 & 36 \\ -21 & 21 & -18 & 80 \end{pmatrix}$$

Determine the Smith Normal Form of A as well as transforming matrices P and Q.
b) Let $\underline{\mathbf{b}} := (-76, 15, 23)^T$. Determine all integer solutions to the system $A\underline{\mathbf{x}} = \underline{\mathbf{b}}$.

2) Let A be a diagonal matrix with entries $d_1, d_2, \ldots d_m$. What is the Smith Normal Form of A?

3) Let $A \in R^{m\times n}$ be a matrix and $S = P \cdot A \cdot Q$ its Smith Normal Form (which is unique). Show that the transforming matrices P and Q are not unique. (Hint: Consider for a square A a centralizing matrix B with $A = B^{-1}AB$.)

4) a) Let $D \in \mathbb{Z}^{n\times n}$ and $\underline{\mathbf{c}} \in \mathbb{Z}^n$. Show that the system $D\underline{\mathbf{y}} = \underline{\mathbf{c}}$ has an integer solution, if and only if for every $\underline{\mathbf{v}} \in \mathbb{Q}^n$, such that $\underline{\mathbf{v}}D$ is an integer vector, (the inner product) $\underline{\mathbf{v}} \cdot \underline{\mathbf{c}}$ is an integer.
b) (This is a theorem due to van der Waerden) Let $A \in \mathbb{Z}^{m\times n}$ and $\underline{\mathbf{b}} \in \mathbb{Z}^n$. Show that the system $A\underline{\mathbf{x}} = \underline{\mathbf{b}}$ has an integer solution, if and only if for every $\underline{\mathbf{u}} \in \mathbb{Q}^n$, such that $\underline{\mathbf{u}}A$ is an integer vector, (the inner product) $\underline{\mathbf{u}} \cdot \underline{\mathbf{b}}$ is an integer.

5*) Let F be a field and $R = F[x]$ the polynomial ring. Take $M \in F^{n\times n}$. We form the characteristic matrix $A = M - x \cdot I$. Let S be the SNF of A over R and $m_M(x)$ be the last nonzero diagonal entry of S (i.e. all other nonzero diagonal entries divide $m_M(x)$). We call $m_M(x)$ the *minimal polynomial* of M. It clearly is a divisor of the characteristic polynomial.
a) Show that every eigenvalue of M is a root of $m_M(x)$.
b) Let $M = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$. Determine $m_M(x)$.
c) Suppose that M has block form $M = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$, where B, C are square matrices. Show that $m_M(x) = \mathrm{lcm}(m_B(x), m_C(x))$.
(With some extra work one can show now that $m_M(x)$ is a generator of the ideal of polynomials, for which $p(M) = 0$.)

6*) Get accustomed with GAP. (Nothing needs to be handed in.)

Problems marked with a *are bonus problems for extra credit.