1) a) Let

\[ A := \begin{pmatrix} 60 & -60 & 48 & -232 \\ -9 & 9 & -6 & 36 \\ -21 & 21 & -18 & 80 \end{pmatrix} \]

Determine the Smith Normal Form of \( A \) as well as transforming matrices \( P \) and \( Q \).

b) Let \( \mathbf{b} := (-76, 15, 23)^T \). Determine all integer solutions to the system \( A \mathbf{x} = \mathbf{b} \).

2) Let \( A \) be a diagonal matrix with entries \( d_1, d_2, \ldots, d_m \). What is the Smith Normal Form of \( A \)?

3) Let \( A \in \mathbb{R}^{m \times n} \) be a matrix and \( S = P \cdot A \cdot Q \) its Smith Normal Form (which is unique). Show that the transforming matrices \( P \) and \( Q \) are not unique. (Hint: Consider for a square \( A \) a centralizing matrix \( B \) with \( A = B^{-1}AB \).)

4) a) Let \( D \in \mathbb{Z}^{n \times n} \) and \( \mathbf{c} \in \mathbb{Z}^n \). Show that the system \( D \mathbf{y} = \mathbf{c} \) has an integer solution, if and only if for every \( \mathbf{v} \in \mathbb{Q}^n \), such that \( \mathbf{v}D \) is an integer vector, (the inner product) \( \mathbf{v} \cdot \mathbf{c} \) is an integer.

b) (This is a theorem due to van der Waerden) Let \( A \in \mathbb{Z}^{m \times n} \) and \( \mathbf{b} \in \mathbb{Z}^n \). Show that the system \( A \mathbf{x} = \mathbf{b} \) has an integer solution, if and only if for every \( \mathbf{u} \in \mathbb{Q}^n \), such that \( \mathbf{u}A \) is an integer vector, (the inner product) \( \mathbf{u} \cdot \mathbf{b} \) is an integer.

5*) Let \( F \) be a field and \( R = F[x] \) the polynomial ring. Take \( M \in F^{n \times n} \). We form the characteristic matrix \( A = M - x \cdot I \). Let \( S \) be the SNF of \( A \) over \( R \) and \( m_M(x) \) be the last nonzero diagonal entry of \( S \) (i.e. all other nonzero diagonal entries divide \( m_M(x) \)). We call \( m_M(x) \) the minimal polynomial of \( M \). It clearly is a divisor of the characteristic polynomial.

a) Show that every eigenvalue of \( M \) is a root of \( m_M(x) \).

b) Let \( M = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \). Determine \( m_M(x) \).

c) Suppose that \( M \) has block form \( M = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \), where \( B, C \) are square matrices. Show that \( m_M(x) = \text{lcm}(m_B(x), m_C(x)) \).

(With some extra work one can show now that \( m_M(x) \) is a generator of the ideal of polynomials, for which \( p(M) = 0 \).)

6*) Get accustomed with GAP. (Nothing needs to be handed in.)

Problems marked with a * are bonus problems for extra credit.