Mathematics 467 **Final (50 points)** Due 5/13/09, 5.40pm

Points (leave blank)					
1	2	3	4	5	$\sum$

Name:

(clearly, please)

This exam is my own work. Sources (apart from the textbooks and my lecture notes) are indicated.

Signature

Notes

- Put your name on this cover sheet and sign it.
- The exam must be given to me, or put in my mailbox (via the front office) by the indicated time.
- You are permitted to use your notes and any publication (book, journal, web page). You are not permitted to consult persons (but you may ask me in class).
  Results which are quoted from a publication (apart from the course textbooks and your lecture notes) should be indicated.
- Unless specified otherwise, you may use a computer for arithmetic. If you are using a longer GAP calculation, just attach a session transcript printout (of the relevant parts. Not more than 2 pages per problem!)
- If you need extra sheets, staple them to this exam. (You do not need to submit scrap paper.)

1) a) Determine the abelian groups of order 48 up to isomorphism.
b) Arrange the following groups according to isomorphism:

$$\mathbb{Z}_2 \times \mathbb{Z}_{12}, \quad \mathbb{Z}_3 \times \mathbb{Z}_8, \quad \mathbb{Z}_4 \times \mathbb{Z}_6, \quad \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4, \quad Z_{24}$$

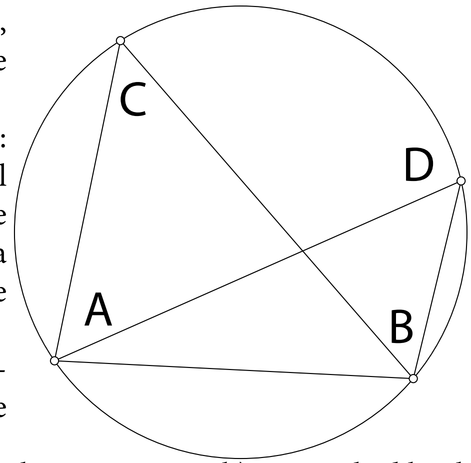
c) Let $G = \mathbb{Z}_{12} \times \mathbb{Z}_4$. How many elements of order 2 does this group have?

2) The equations

$$x = 3\cos(t) - \cos(3t), \qquad y = 3\sin(t) - \sin(3t)$$

define a parametric curve (called a *Nephroid* – a kidney shaped curve).
a) Rewrite the equations as polynomial equations involving x, y, $s = \sin(t)$, and $c = \cos(t)$.
b) Using that $s^2 + c^2 = 1$, determine a (polynomial) equation in (only) x and y describing the curve. (Rewrite the equations as polynomials, and eliminate t.) (**Hint:** The equation will be nicer to write down, if you consider $(x^2 + y^2 - 1)^3$ as a part.)

3) Suppose the (four, different) points A, B, C, D lie on a circle. We want to show that the angle ACD equals the angle ADB.



a) Assume the points are given by coordinates: $A = (a, b)$ and so on. Write down polynomial equations that indicate that D lies on the circle through A, B, C (**Hint:** You want to introduce a further point $E = (x_1, x_2)$ for the center of the circle.)

b) Write down polynomial equations that indicate equality of the angle ACD and the angle ADB.

c) Show (using Gröbner bases for an ideal) that the equations in b) are implied by the equations in a).

4) Let F be a field. Prove that if $[F(\alpha) : F]$ is odd, then $F(\alpha) = F(\alpha^2)$.

5) Let E be the splitting field of $x^9 - 1$ over $\mathbb{Q}$. (We know from problem 61, that $[E : \mathbb{Q}] = \varphi(9) = 6$.)

a) Determine $G := \mathrm{Gal}(E/\mathbb{Q})$.

b) Determine the subgroups of G.

c) For every subgroup $S \leq G$, determine the corresponding fixed field $F := \mathrm{Fix}(S)$.

d) For every subfield F obtained this way, determine $\mathrm{Gal}(F/\mathbb{Q})$.