Mathematics 466	**Midterm (50 points)**	10/9/19

Points (leave blank)					
1	2	3	4	5	$\sum$

Name:

(clearly, please)

Honor pledge: I have not given, received, or used any unauthorized assistance.

____

Signature

You may use a pocket calculator that is incapable of transmitting data; it may not store user-defined information. You also may bring a handwritten single page, letter size with notes. You can work on the problems in any order you like. Show your work! All problems carry the same weight. Justifications are a crucial part of a solution.

1) Let G be a group and $S \leq G$ a subgroup. We define a subgroup

$$N = \{g \in G \mid g^{-1}Sg = S\}.$$

Show that N is a subgroup of G (it is called the normalizer). (**Hint:** You can either use the subgroup test, or observe that N is a stabilizer under a suitable (say which) action.)

Let $a, b \in N$. Then $(ab)^{-1}Sg = b^{-1}\underbrace{a^{-1}Sa}_{=S,\text{ since } a\in N}b = b^{-1}Sb = S$

Thus $ab \in N$. Ditto $b^{-1}Sb = S \Rightarrow S = bSb^{-1} \Rightarrow b^{-1} \in N$.

Finally $N \neq \emptyset$, as $1_G \in S$, as $1^{-1}S1 = S$.

Thus $S \leq G$

2) Let $G = \mathbb{Q} - \{0\}$ the multiplicative group of nonzero rational numbers and

$$H = \left\{ \begin{pmatrix} a & a - 1/a \\ 0 & 1/a \end{pmatrix} \mid a \in \mathbb{Q}, a \neq 0 \right\} \leq \mathrm{SL}_2(\mathbb{Q})$$

(You do not need to show that it is a subgroup.) Show that G is isomorphic to H.

Let $\varphi : G \to H$, $a \mapsto \begin{pmatrix} a & a - \frac{1}{a} \\ 0 & 1/a \end{pmatrix}$.

Then $\varphi(a \cdot b) = \begin{pmatrix} ab & ab - \frac{1}{ab} \\ 0 & \frac{1}{ab} \end{pmatrix} = \begin{pmatrix} a & a - \frac{1}{a} \\ 0 & \frac{1}{a} \end{pmatrix} \begin{pmatrix} b & b - \frac{1}{b} \\ 0 & \frac{1}{b} \end{pmatrix}$,

$= \varphi(a)\varphi(b)$, so φ is homomorphism.

φ is onto, as any matrix $\begin{pmatrix} a & a - 1/a \\ 0 & 1/a \end{pmatrix} = \varphi(a)$

$\ker(\varphi) = \left\{ a \mid \begin{pmatrix} a & a - 1/a \\ 0 & 1/a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} = \{1\}$

Thus φ is one-to-one, and thus an isomorphism.

3) Let $G = \langle g \rangle$ be a cyclic group and $S \leq G$.
a) Show that $S \triangleleft G$, and that G/S is cyclic.
b) Assuming that $|G| = n$ and $|S| = k$ (dividing n), find a set of representatives for the cosets of S in G. (**Hint:** First note how many cosets there are.)

a) Let $x \in S$ and $y \in G$. Then $x = g^a$ and $y = g^b$ and $y^{-1}xy = g^{-b}g^a g^b = g^a = x$, so $S \triangleleft G$.

If $x \in G/S$ then $x = S \cdot g^a$ for suitable a, thus G/S is generated by Sg, and thus cyclic.

b) There will be $\frac{n}{k}$ many cosets. Since $G/S = \langle Sg \rangle$, they must be of the form Sg^a for $a = 0, 1, \ldots$, thus reps are $g^0, g^1, \ldots, g^{\cancel{b}\,k-1}$

4) Consider three coins, lying in a row, heads-up. We label their heads as 1,3,5, their tails respectively as 2,4,6. We now consider symmetry operations that consist of turning individual coins over, or swapping the positions of coins. These operations generate a group, we describe it by permutations which come from the action on the numbers, and get the group (generators are individual coin flips and swaps of adjacent coins)

$$G = \langle (1,2), (3,4), (5,6), (1,3)(2,4), (3,5)(4,6) \rangle,$$

Let $S = \mathrm{Stab}_G(6)$ and $T = \mathrm{Stab}_S(4) = \mathrm{Stab}_G(6) \cap \mathrm{Stab}_G(4)$ the subgroups of operations that fix the right, respectively the right and middle coin. (You may assume everything up to now as given and do not need to show any of it.)

a) What is $|T|$? (**Hint:** The action of T can only change the left coin.)

b) Calculate the orbit of 4 under S. Use this to calculate $|S|$ from $|T|$.

c) Calculate the orbit of 6 under G. Use this to calculate $|G|$ from $|S|$.

d) Give an example of a permutation of S_6 that is not in G. Explain why.

a) 2 (flip or not)

b) Orbit is $\{1,2,3,4\}$, thus $|S| = 4 \cdot 2 = 8$

c) Orbit is $\{1,2,3,4,5,6\}$, thus $|G| = 6 \cdot 8 = 48$.

d) $(1,3)$ would have to destroy the two leftmost coins, so $(1,3) \notin G$.

5) Let G be a group and $U \le G$ with $[G:U] = 2$ and $x \in G$ with $x \notin U$. Then U and Ux are the two cosets of U. We have seen in class that G acts on the cosets of U by right multiplication. (you do not need to show any of this).

a) Let $g \in U$. Show that $Ug = U$ and $Uxg = Ux$ (that is g fixes both cosets).
b) Let $g \in G$ but $g \notin U$. Show that $Ug = Ux$ and $Uxg = U$ (that is g swaps the two cosets). (**Hint:** $Uxg = U$ is equivalent to $Ux = Ug^{-1}$.)
The results of a) and b) together show that the map

$$\varphi\colon G \to S_2 = \langle(1,2)\rangle, \qquad g \mapsto \begin{cases} () & \text{if } g \in U \\ (1,2) & \text{if } g \notin U \end{cases},$$

is a homomorphism (you do not need to show this.)
c) Show that $\ker\varphi \le U$.
d) How large is the image of φ? Conclude that $U = \ker\varphi$ and thus that $U \lhd G$.
(You thus have shown that a subgroup of index 2 is normal, as was claimed, but not proven, in class.)

a) If $g \in U$ then $Ug = U$. Since G permutes the cosets we can't have $Uxg = U$ as well, thus $Uxg = Ux$

b) As $g \notin U$, $Ug \neq U$. But there are only two cosets, thus $Ug = Ux$. Also $g^{-1} \notin U$, thus $Ug^{-1} = Ux \Rightarrow U = Uxg$.

c) Let $g \in \ker\varphi$. Then $Ug = U$ (as $\varphi(g) = ()$), thus $g \in U$.

d) We know that $\varphi(x) \neq ()$, thus $\varphi(G) \neq \{()\}$, thus $\varphi(G) = S_2$ and thus $[G:\ker\varphi] = 2$ by the homomorphism theorem $\Rightarrow U = \ker\varphi$.