27) Let \( R \) be a ring, \( I \subset R \) a prime ideal and \( \nu: R \to R/I \) the natural homomorphism. We define \( \theta: R[x] \to (R/I)[x] \) by \( \sum a_i x^i \mapsto \sum (I + a_i)x^i \in (R/I)[x] \).

a) Show that \( \theta \) is a ring homomorphism.

b) Let \( f \in R[x] \) be monic (i.e. leading coefficient 1). Show that the degree distribution of a factorization of \( f \theta \) is a refinement of the degree distribution of a factorization of \( f \). (In particular if \( f \theta \) is irreducible, then also \( f \) is irreducible.)

Let's look at this in an example. We set \( R = \mathbb{Z} \) and consider \( I = \langle p \rangle \subset \mathbb{Z} \). The following example shows, how to evaluate \( \theta \) in GAP:

```gap
gap> x:=X(Rationals,"x");;
gap> f:=x^4+3*x+1;; # some polynomial

gap> r:=Integers mod 2;; # factor ring
gap> y:=X(r);; # create a polynomial variable over the factor ring.
gap> fr:=Value(f,y); # this evaluates f theta
x_1^4+x_1+Z(2)^0

gap> List(Factors(fr),Degree); # 1 factor of degree 4
[ 4 ]
```

We see that \( f \) is irreducible mod 2 and therefore irreducible over \( \mathbb{Z} \).

For another example, let \( f = x^3 + rx^2 + 9x^2 + 31 \). Then \( f \) reduces modulo 2 in two factors of degree 3 and modulo 3 in two factors of degree 2. The only way both can be refinements of the degree distribution over \( \mathbb{Z} \) is if \( f \) is irreducible.

c) Show that the following polynomials \( \in \mathbb{Z}[x] \) are irreducible:

1. \( x^5 + 5x^2 - 1 \)
2. \( x^4 + 4x^3 + 6x^2 + 4x - 4 \)
3. \( x^7 - 30x^5 + 10x^3 - 120x + 60 \)

(Does this always work? No — one can show that there are irreducible polynomials, for example \( x^4 - 10x^2 + 1 \), which modulo \( every \) prime splits in factors of degree 1 or 2.)

28' \) (GAP) The GAP list Primes contains the 168 prime numbers \( < 1000 \). Determine the degree distributions (and their frequencies) when factoring \( x^4 + 4x^2 + 4x - 4 \) modulo these primes. How frequently (roughly) does each degree distribution occur? (You might want to concentrate on the frequency of primes for which the polynomial will split into linear factors.) Repeat the experiment for \( x^4 - 10x^2 + 1 \) and \( x^4 + 3x + 1 \). What frequencies occur in these cases?

29) Show that a finite integral domain is a field.

30) Let \( R = \mathbb{Z}[^{\sqrt{-5}}] \) (This ring does not have unique factorization and therefore cannot be a UFD!) and let \( I = \langle 2, 1 + \sqrt{-5} \rangle, J = \langle 3, 2 + \sqrt{-5} \rangle \) and \( K = \langle 3, 2 + \sqrt{-5} \rangle \).

a) Show that \( I \subset R \) is not principal, i.e. there is no \( x \in R \) such that \( I = \langle x \rangle \).

b) Show that \( I^2 = \langle 2 \rangle \).

(i.e. show that every product \( (a \cdot 2 + b(1 + \sqrt{-5}) (c \cdot 2 + d(1 + \sqrt{-5}) \) is a multiple of 2, and that 2 can be written as sum of such products.)

c) Show (this is similar to b), that \( JI = \langle 1 - \sqrt{-5} \rangle \) and \( IK = \langle 1 + \sqrt{-5} \rangle \). Conclude that \( I^2 J K = \langle 6 \rangle \).

**What is this good for:** The factorization \( 2 \cdot 3 = 6 \) (or — on ideal level — \( (2) \cdot (3) = (6) \)) thus is not a factorization into irreducible ideal, but one has that \( (2) = I^2 \) and \( (3) = JK \). The alternative factorization \( 6 = 1 + \sqrt{-5} (1 - \sqrt{-5}) \) then on ideal level splits up as product of the same constituents in different order \( IJK \). One can show that this factorization of ideals is unique, this property gave rise to the name "Ideal", as "ideal numbers".