24) a) If $q = p_1 \cdot \dots \cdot p_k$ is a product of different primes (i.e. no p_i^2 divides q), show that every **abelian** group of order G is isomorphic to the cyclic group C_q of order q. (You may assume, without proof, that the group has elements of order p_i for every pime p_i.)
b) Show that a group of order 4 must be abelian and will be isomorphic to either C_4, or to $V_4 = \langle (1,2)(3,4), (1,3)(2,4) \rangle$.
c) Show that every group of order 6 is isomorphic to either C_6, or the group S_3.

25) [This problem is slightly harder, but the difficulty is mainly in juggling the different kinds of objects. In each part, think about what you need to show, going back to the definitions. Then you will have solved already 3/4 of the problem.]
a) Let G be a group and $x \in G$ a fixed element. Show that the map $\theta_x: G \to G$, $g \mapsto x^{-1}gx$ is an isomorphism from G to G.
b) Show that the set

$$\mathrm{Aut}(G) = \{\theta: G \to G \mid \theta \text{ is an isomorphism}\}$$

is a group, with composition of mappings as operation.
c) Show that the map

$$\alpha: G \to \mathrm{Aut}(G), x \mapsto \theta_x$$

(with θ_x defined as in a) is a group homomorphism.
d) Show that $\mathrm{Image}(\alpha) \lhd \mathrm{Aut}(G)$. (This subgroup is called the group of inner automorphisms.)
e) Show that

$$\ker \alpha = Z(G) = \{x \in G \mid xg = gx \forall g \in G\}$$

26) Let $D_8 = \langle (1,2,3,4), (1,3) \rangle$ be the dihedral group of order 8. We have classified the subgroups of D_8 in class as:

$$\begin{array}{l} \langle () \rangle, \langle (1,3) \rangle, \langle (2,4) \rangle, \langle (1,3)(2,4) \rangle, \langle (1,4)(2,3) \rangle, \\ \langle (1,2)(3,4) \rangle, \langle (1,3),(2,4) \rangle, \langle (1,2)(3,4),(1,3)(2,4) \rangle, \\ \langle (1,2,3,4) \rangle, D_8 \end{array}$$

Which of these subgroups are normal? Describe their factor groups. What groups are possible images of D_8 under homomorphisms? Does D_8 have any nonabelian homomorphic images?

27) Let $G = \mathrm{GL}_n(p)$ the group of $n \times n$ invertible matrices with entries modulo p. Let $\mathrm{SL}_n(p)$ the group of matrices of determinant 1 and S the group of scalar matrices (multiples of the identity).
a) Show that $\mathrm{SL}_n(p) \lhd G$ and $S \lhd G$.
b) Show that $G/\mathrm{SL}_n(p) \cong U(p)$. (Hint: Use the homomorphism theorem: What would be a suitable homomorphism?.)

28) Let $G \leq S_n$ be a permutation group and $\Omega = \{\{a, b\} \mid 1 \leq a \neq b, \leq n\}$ the class of all 2-element subsets of numbers in $\{1, \ldots, n\}$ (i.e. the set $\{a, b\}$ is considered the same as $\{b, a\}$). We know that $|\Omega| = \binom{n}{2}$. We define an action μ of G on Ω by acting on the entries:

$$\mu(\{a, b\}, g) := \{a^g, b^g\}$$

(where a^g is the usual image of a point a under a permutation g.)
a) Show that this defines a group action.
b) For $G = D_8 = \langle(1, 2, 3, 4), (1, 3)\rangle$, determine the orbits of G on Ω.

29) A dodecahedron is a regular solid with 12 faces, each of which is a regular pentagon. At each corner three pentagons meet. We consider symmetries given by rotating the dodecahedron in space.
a) How long is the orbit of one face?
b) How large is the stabilizer of one face?
c) Determine the number of rotational symmetries from the results in a) and b).
d) Repeat the calculation for orbit and stabilizer of one of the 20 corners.