51) Let \( G = \langle (1, 2, 3), (2, 3, 4) \rangle \). The following GAP command determines all subgroups of \( G \):
\[
s := \text{Union}(\text{List}((\text{ConjugacyClassesSubgroups}(G)), \text{Elements}));
\]
Draw a subgroup lattice (as done in the lecture for the symmetries of a square). Which of the subgroups are cyclic? Which ones are normal. Does \( G \) have a subgroup of order \( k \) for every \( k \) dividing \( |G| \)? (Useful GAP commands: \text{IsSubset}(S, T), \text{IsCyclic}(S), \text{IsNormal}(G, S).)

52) a) Let \( G \) be a group and \( x \in G \) a fixed element. Show that the map \( \theta_x: G \to G, g \mapsto x^{-1}gx \) is an isomorphism from \( G \) to \( G \).
b) Show that the set
\[
\text{Aut}(G) = \{ \theta: G \to G \mid \theta \text{ is an isomorphism} \}
\]
is a group, with composition of mappings as operation.
c) Show that the map
\[
\alpha: G \to \text{Aut}(G), x \mapsto \theta_x
\]
(with \( \theta_x \) defined as in a) is a group homomorphism.
d) Show that \( \text{Image} \alpha \lt \text{Aut}(G) \). (This subgroup is called the group of inner automorphisms.)
e) Show that
\[
\ker \alpha = \{ x \in G \mid xg = gx \forall g \in G \}
\]
(This subgroup is called the \textit{Center of } \( G \) and is denoted — because of the German name “Zentrum” — by \( Z(G) \).)

53) Describe (e.g. find a set of representatives) the cosets of \( S_{n-1} \) in \( S_n \).

54*) Let \( G \leq S_n \) be a permutation group and \( \Omega = \{ \{ a, b \} \mid 1 \leq a \neq b \leq n \} \) the class of all 2-element subsets of numbers in \{1, \ldots, n\} (i.e. the set \{a, b\} is considered the same as \{b, a\}). We know that \( |\Omega| = \binom{n}{2} \). We define an action \( \mu \) of \( G \) on \( \Omega \) by acting on the entries:
\[
\mu (\{ a, b \}, g) := \{ a^g, b^g \}
\]
(where \( a^g \) is the usual image of a point \( a \) under a permutation \( g \).)
a) Show that this defines a group action.
b) For \( G = D_8 = \langle (1, 2, 3, 4), (1, 3) \rangle \), determine the orbits of \( G \) on \( \Omega \).

55) A dodecahedron is a regular solid with 12 faces, each of which is a regular pentagon. At each corner three pentagons meet. We consider symmetries given by rotating the dodecahedron in space.
a) How long is the orbit of one face?
b) How large is the stabilizer of one face?
c) Determine the number of rotational symmetries.
d) The dodecahedron has 6 axes which connect the midpoints of opposite faces. The group of symmetries acts on these axes. Show that this action is faithful (i.e. that the identity is the only element fixing all the axes). (\textbf{Hint: } Every rotation is a linear transformation and the axes span the space.)
e) Write down permutation generators for the group of rotations. (Write down permutations for two or three rotations around different axes, then use GAP to show that the group generated by them has the order as calculated in c.)

Problems marked with a * are bonus problems for extra credit.