Mathematics 466 **Homework (due Sep. 6)** A. Hulpke

1) Which of the following sets are groups? Either show that the axioms are fulfilled or show which axiom is violated. If the set forms a group, is it abelian?

a) Even integers under addition.

b) Even integers under multiplication.

c) Odd integers under addition.

d) $\{z \in \mathbb{C} \mid |z| = 1\}$ under ordinary multiplication.

e) $\{1, 2, 3\}$ under multiplication modulo 4.

f) $\{0, 1, 2, 3, 4\}$ under multiplication modulo 5.

g) The set $\mathbb{Q}^+ = \{a \in \mathbb{Q} \mid a > 0\}$ with the operation

$$a \cdot b = \text{take } a \text{ percent of } b.$$

h) The set of all matrices in $\mathbb{R}^{3\times 3}$ of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \qquad a, b, c \in \mathbb{R}$$

under matrix multiplication.

i) For a given real number c, the numbers in $(-c, c)$ with operation (this is the addition of velocities in special relativity)

$$x \circ y = \frac{x + y}{1 + xy/c^2}$$

Hint: You do not have to prove again statements we have proven more generally in the lecture (for example: Addition and Multiplication of complex numbers is associative, matrix multiplication is associative), but can just refer to it.
Do not forget to check that the set is closed under the operation and that inverse elements are indeed in the set!

2) Let G be a (not necessarily abelian!) group and $a, b \in G$.
a) Show that $(a^{-1})^{-1} = a$. (**Hint:** You have to show that $x = a$ fulfills the defining condition: $xa^{-1} = e$, $a^{-1}x = e$.)
b) Show that $(ab)^{-1} = b^{-1}a^{-1}$.
c) Give an example of a group G and elements a, b, such that $(ab)^{-1} \neq a^{-1}b^{-1}$.
d) Show that $((ab)^2)^{-1} = b^{-1}a^{-1}b^{-1}a^{-1}$.

3) Show that the following multiplication table cannot be the Cayley table for a group:

$$\left(\begin{array}{c|ccccc} & a & b & c & d & e \\ \hline a & a & b & c & d & e \\ b & b & a & d & e & c \\ c & c & d & e & a & b \\ d & d & e & b & c & a \\ e & e & c & a & b & d \end{array}\right)$$

4) Let G be a group such that for every $g \in G$ we have that $g^2 = 1$. show that G must be abelian. Give an example of such a group for which $|G| > 2$.

5) For a group element $g \in G$ and an integer n, we define the n-th power (in the obvious way) as

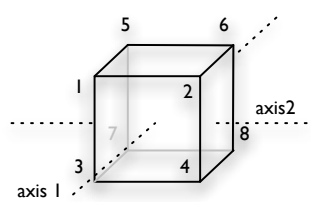
$$g^n = \begin{cases} \underbrace{g \cdot \cdots \cdot g}_{n \text{ factors}} & \text{if } n > 0 \\ \underbrace{g^{-1} \cdot \cdots \cdot g^{-1}}_{-n \text{ factors}} & \text{if } n < 0 \\ 1 & \text{if } n = 0 \end{cases} .$$

For any $g \in G$ we define the *order* of g as

$$|g| = \min\{n \mid n > 0 \text{ and } g^n = 1\}.$$

(with $|g| = \infty$ if there is no such n).
Show that for $a, b \in G$ we have that $|ab| = |ba|$.

6) Consider the following cube, whose corners we'll label with the numbers 1 to 8:



Write down permutations in cycle form that show how the corners change under the following operations.
a) Clockwise rotation by 90° around the axis (axis 1) that goes through the front and back side.
b) Rotation (by 180°) around the axis (axis 2) going through the right and left side.
c) Does the permutation $(1,2)(3,4)(5,6)$ correspond to a symmetry of the cube? Why (not)?