In this note, we want to determine a strategy for solving the $2 \times 2 \times 2$ Rubik's cube by hand. (Again, similar approaches work for other puzzles, this one is just small enough to be feasible in class.)
Remember that we label the faces of the cube as shown and fix the piece (16/19/24) in space. The group of symmetries then is

$$G = \langle \quad \text{top} = (1, 2, 4, 3)(5, 17, 13, 9)(6, 18, 14, 10),$$
$$\text{left} = (1, 9, 21, 20)(5, 6, 8, 7)(3, 11, 23, 18),$$
$$\text{front} = (3, 13, 22, 8)(4, 15, 21, 6)(9, 10, 12, 11) \quad \rangle$$
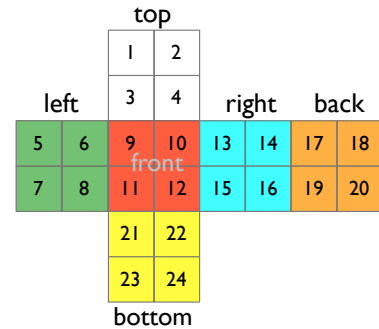
The basic idea now is to place pieces in the right position (e.g. in "layers" of the cube), and then continue with movements that leave these correctly placed pieces unmoved. Moving the first pieces is usually easy; the hard work is to find suitable elements in the stabilizer or the positions filled so far.

We will choose the positions to be filled in a way that guarantees that many of the original generators fix the chosen points. This will automatically make for some stabilizer generators being represented by short words. Concretely, we will fill the positions starting in the order (23,22,21) and then work on the top layer.

We start by computing the orbit of 23. This is done in a similar way as we would enumerate all group elements: We calculate the image of all points processed so far under all generators, until no new images arise. We also keep track of group elements which map 23 to the possible images.

It is possible to do these slightly tedious calculations with GAP. The following set of commands will calculate the orbit and keep track of words that yield the same image. (We work with words, as they are easier to read than images.)

```
map:=EpimorphismFromFreeGroup(cube:names:=["T","L","F"]);
gens:=GeneratorsOfGroup(cube);
letters:=GeneratorsOfGroup(Source(map)); # corresponding letters
orb:=[23]; # the orbit being built
words:=[One(letters[1])]; # words that produce the right images
for x in orb do
  for i in [1..Length(gens)] do
    img:=x^gens[i];
    if not img in orb then
      Add(orb,img);
      p:=Position(orb,x);
      Add(words,words[p]*letters[i]);
    fi;
  od;
od;
```

As a result we get the following orbit and representatives:

Orbit	23	18	14	3	10	1	11	13	6	12	2
Rep	e	L	LT	L^2	LT^2	L^2T	L^3	L^2F	LT^3	LT^2F	L^2T^2

Orbit	9	22	8	4	5	21	7	15	17	20
Rep	L^2TL	L^2F^2	LT^3L	LT^3F	L^2TLT	L^2TL^2	LT^3L^2	LT^3F^2	L^2TLT^2	L^2TL^3

The crucial idea for stabilizer elements (apart from the obvious ones, such as top rotation here) now is the following: Suppose that the image of an orbit element x under a generator g gives an *old* orbit element y. Then $\text{Rep}_x \cdot g \cdot \text{Rep}_y^{-1}$

moves the starting element (here 23) to itself and therefore lies in the stabilizer. (One can show[1], that all of these elements generate the stabilizer.)

In our example, we find that $23^T = 23$, so $e \cdot T \cdot e^{-1} = T$ must lie in the stabilizer. Similarly $11^F = 9$, so $L^3 \cdot F \cdot (L^2 TL)$ lies in the stabilizer, and so on.

By computing group orders we find that we just need a few elements for generation, namely

$$C_1 := \text{Stab}_G(23) = \langle T, F, L^{-1}TL \rangle$$

(It is worth noticing that the construction process tends to produce elements of the form ABA^{-1}. If you consult any printed solution strategy, this kind of shape indeed abounds.) As far as solving the cube is concerned, it is very easy to get a few initial positions right. Therefore we don't aim for a systematic strategy yet, but keep the stabilizer generators for the next level.

The next position we work with is 22, similarly as above (again, there is a nontrivial calculation involved in showing that these three elements suffice, if one takes just a few random elements there is no a-priori guarantee that they would do) we get

$$C_2 := \text{Stab}_G(22, 23) = \langle T, L^{-1}TL, FTF^{-1} \rangle$$

Next (this is completing the bottom layer), we stabilize 21. This is where the stabilizer becomes interesting, as far as continuing to solve the puzzle is concerned. A repeated application gets

$$C_3 := \text{Stab}_G(21, 22, 23) = \langle T, LT^{-1}L^{-1}T^{-1}F^{-1}LF \rangle$$

The second generator here is chosen amongst many other possibilities as being of shortest possible length. Its action on the cube is the permutation $(1, 17, 5, 14, 18, 2)(4, 10, 13)$, i.e. two pieces are swapped (and turned) and one piece is turned. If we only consider the positions (but not rotations) of the top four pieces, it is not hard to see (homework) that together with the top rotation this will let us place every piece in its right position up to rotation. (This means, we are stabilizing the *sets* $\{1, 5, 18\}$, $\{2, 14, 17\}$, $\{3, 6, 9\}$ and $\{4, 10, 13\}$ — formally we are considering a different group action here, namely one on the cubies.)

$(1,17,5,14,18,2)(4,10,13)$

What is left now is to orient the top 4 pieces suitably. By looking through stabilizer generators we find the element $L^{-1}FL^{-1}FT^{-1}L^{-1}T^2F^2LT$ with the permutation $(3, 6, 9)(4, 13, 10)$. It rotates the top two front cubies in opposite directions. Clearly similar movements are possible to the top cubies on the other sides, by rotating the whole cube in space. By applying these moves for front, left and back, we can determine the placement of three of the top cubies arbitrarily.
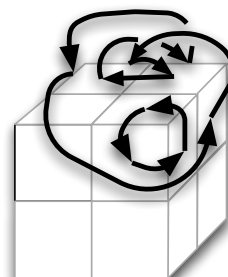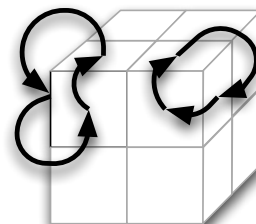
$(3,6,9)(4,13,10)$

We now claim that this will actually solve the cube completely. We know that $[G : C_1] = |\text{orb}_G(23)| = 24 - 3 = 21$ (on the whole cube, the piece in position 23 can be moved in all positions, short of the three fixed ones). Similarly, when stabilizing position 23, we can (this can be easily checked by trying out the moves permitted in C_1) move the piece in position 22 to all places, but the 3 fixed ones (16/19/24) and the three that share a cubie with 23 (7/20/23), so $[C_1 : C_2] = 18$. By a similar argument $[C_2 : C_3] = 15$.

Under the stabilizer C_3 we have seen that we can correctly place the four cubies in all possible permutations, thus the group stabilizing the cubie positions has index 24 in C_3. Using that $|G| = 3674160$, we thus get that the group which only turns (but not permutes) the top cubies, and leaves the bottom layer fixed has order

$$\frac{3674160}{21 \cdot 18 \cdot 15 \cdot 24} = 27 = 3^3$$

But this means that turning only three cubies correctly must also place the fourth cubie in the right position.

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[1]This is called Schreier's lemma, see for example Holt: Handbook of Computational Group Theory