

13) a) Compute $5^{27} \pmod{7}$.

b) Solve (find all solutions) the following linear congruences by hand:

$$11x \equiv 28 \pmod{37}, \quad 42x \equiv 90 \pmod{156}$$

In this problem you may use a computer do division with remainder, but your solution should show some intermediate steps and not only quote a computer result.

14) Prove that if $a \equiv b \pmod{m}$, then $(a, m) = (b, m)$.

15) Three sailors and a monkey end up on a tropical beach. They collect n coconuts for food and decide to divide them up in the morning. In the night, the first sailor gets up, divides the coconuts into three equal parts, which leaves a remainder of one, and gives the one remaining coconut to the monkey. He takes his share, puts the remaining pile together, and leaves.

Later the second sailor wakes up. Without noticing that the first sailor already left, he divides the coconuts by three, gives the one remaining coconut to the monkey.

Finally the third sailor repeats the process.

Find the smallest possible (positive – the theoretical physicist P.A.M. DIRAC allegedly gave the answer -2) number n of coconuts in the original pile.

(Hint: Let x, y, z be the numbers of coconuts each sailor took. Then $n = 3x + 1$, $2x = 3y + 1$, $2y = 3z + 1$. This gives a linear diophantine equation between z and n .)

16)* Let $n = \sum_{i=0}^k a_i 10^i$ ($0 \leq a_i \leq 9$) be a (positive) integer in decimal expansion.

Show that $9|n$ if and only if $9|\sum_{i=0}^k a_i$.

(For example, $9|1 + 5 + 3$, so $9|153$.)

17) Ernie, Bert and the Cookie Monster want to measure the length of Sesame Street. Each of them does it his own way. Ernie relates: "I made a chalk mark at the beginning of the street and then again every 7 feet. there were 2 feet between the last mark and the end of the street." Bert tells you: "Every 11 feet there are lamp posts in the street. The first is 5 foot from the beginning and the last one is exactly at the end of the street." Finally the Cookie Monster says: "starting at the beginning of Sesame Street, I put down a cookie every 13 feet. I ran out of cookies 22 feet from the end." All three agree that the length does not exceed 1000 feet. How many feet is Sesame Street long?

(You may use the computer to compute gcds, but the solution should not only involve a call to the ChineseRemainder function.)

18) Suppose you work with a (very old) computer that only works with 1-digit decimal numbers. You want to compute the determinant d of the matrix

$$\begin{pmatrix} -291 & -84 & 28 \\ 1015 & 293 & -97 \\ 21 & 6 & -4 \end{pmatrix}$$

and are given the information that the absolute value $|d|$ is smaller than 15.

Compute the determinant modulo different small primes (for example 2,3 and 5) and use the Chinese Remainder Theorem to obtain the proper value of d .

(You may use the ChineseRemainder command in GAP.)