We have seen that the test for spanning or linear independenced of vectors in $\mathbb{F}^n$ consists of solving a system $A\underline{\mathbf{x}} = \underline{\mathbf{b}}$ where the columns of A are given by the vectors. On this sheet we want to collect this information.

Suppose we are given $A \in \mathbb{F}^{m\times n}$. We can consider the columns of A as vectors $\underline{\mathbf{v_1}}, \ldots, \underline{\mathbf{v_n}} \in \mathbb{F}^m$, thus having $A = \left(\underline{\mathbf{v_1}} | \cdots | \underline{\mathbf{v_n}}\right)$. On the other hand, we can consider the rows of A as (transposed) vectors $\underline{\mathbf{r_1}}, \ldots, \underline{\mathbf{r_m}} \in \mathbb{F}^n$, thus having $A = \begin{pmatrix} \underline{\mathbf{r_1}} \\ \hline \vdots \\ \hline \underline{\mathbf{r_m}} \end{pmatrix}$.

Definition The row space of A is the subspace of $\mathbb{F}^n$ spanned by the rows of A: $\mathrm{RS}(A) = \mathrm{Span}(\underline{\mathbf{r_1}}, \ldots, \underline{\mathbf{r_m}}) \leq \mathbb{F}^n$.
The column space of A is the subspace of $\mathbb{F}^m$ spanned by the columns of A: $\mathrm{CS}(A) = \mathrm{Span}(\underline{\mathbf{v_1}}, \ldots, \underline{\mathbf{v_n}}) \leq \mathbb{F}^m$.
Using this notation, we know already that:

- The expression $A \cdot \underline{\mathbf{x}}$ is a linear combination of the $\underline{\mathbf{v_i}}$ with coefficients given by the entries of $\underline{\mathbf{x}}$, and thus an element of $\mathrm{CS}(A)$. Thus:
- The system $A\underline{\mathbf{x}} = \underline{\mathbf{b}}$ has a solution if and only if $\underline{\mathbf{b}} \in \mathrm{CS}(A)$.
- The columns of A are linear independent, if and only if the system $A\underline{\mathbf{x}} = \underline{\mathbf{0}}$ has only the trivial solution $\underline{\mathbf{x}} = \underline{\mathbf{0}}$.
- Also remember that $N(A) = \{\underline{\mathbf{x}} \mid A\underline{\mathbf{x}} = \underline{\mathbf{0}}\}$. Thus $N(A) = \{\underline{\mathbf{0}}\}$ if and only if the columns of A are linearly independent.

Row transformations

On the other hand, the row space can be convenient when computing a basis:

Lemma: Elementary row operations on A do not change $\mathrm{RS}(A)$.

Lemma: If A is (transformed) into row echelon form, the nonzero rows of A form a basis of $\mathrm{RS}(A)$.

Corollary: The dimension of $\mathrm{RS}(A)$ is equal to the number of nonzero rows in the REF of A.

Corollary: $\dim(\mathrm{RS}(A)) + \dim(N(A)) = n$ (Nr. of columns)
What happens to the column space if we do row transformations? The space can clearly change, as the example $A = \mathrm{Span}((1,0)^T)$ shows. However (that's how we compute the nullspace!) row transformations do not change the nullspace $N(A)$.

Lemma: If we delete columns in A which are nonstep in the REF of A and obtain a matrix B, the REF of B is obtained from the REF of A by deleting these same columns.

Corollary: A basis for $\mathrm{CS}(A)$ is given by taking the columns of A (not of REF(A)) whose positions are step columns in the REF of A.

Corollary: $\dim(\mathrm{CS}(A)) = \dim(\mathrm{RS}(A)) = n - \dim(N(A))$. We call this dimension the rank of A.