33) Let $V = \mathbb{R}^2$ with the standard basis $S$ and $L: \mathbb{R}^2 \to \mathbb{R}^2$ given by $\overline{s} [L]_S = \begin{pmatrix} -3 & -8 \\ 10 & 27 \end{pmatrix}$.

a) What is $L((5,3)^T)$?

b) Let $B = \{(1,1)^T, (2,3)^T\}$. Determine $\overline{b} [L]_B$.

c) For $y = B \ast (3,4)^T$, compute $[L(y)]_B$.

34) Let $L: \mathbb{P}_4 \to \mathbb{R}^2$, $f(x) \mapsto (f(1),f(2))^T$.

a) For $B = \{1,x,x^2,x^3\}$ and $S$ the standard basis of $\mathbb{R}^2$, determine $\overline{s} [L]_B$.

b) Determine bases for $N(L)$ and $\text{Im}(L)$.

35) a) Let $V = \mathbb{R}^2$ with the standard basis $S = \{(1,0)^T, (0,1)^T\}$. A linear transformation $L: V \to V$ transforms the picture 1. below to the picture 2. Write down the matrix $\overline{s} [L]_S$.

(\textbf{Hint:} Consider how corner points transform and use this to deduce what the images of the basis vectors are.)

![Pictures 1 to 7](image)

b) Repeat the process in a) to find matrices transforming picture 1. into 3., 4., 5., 6. and 7. subsequently.

c) Find a matrix for the transformation of picture 2 into picture 5.

d) How does the sign of the determinant of the transformation matrix relate to the graphical transformation?

36) Let $V = \mathbb{P}_3$ and $L: V \to V$ defined by

$$L(f(x)) = x \cdot f'(x) + f''(x)$$

a) Let $B = \{1,x,x^2\}$ and $\mathcal{C} = \{1,x,1+x^2\}$. Determine $\overline{c} [\text{id}]_B$.

b) Determine $\overline{c} [L]_\mathcal{C}$ and $\overline{b} [L]_B$. 
c) For \( f(x) = a + bx + cx^2 \), determine a formula for \( L^n(f(x)) = L(L(\cdots L(f(x)))) \) for all positive integers \( n \).

**Hint:** Consider the problem in basis \( \mathcal{C} \).

### 37

Let \( M, N \in \mathbb{F}^{n \times n} \) and \( B \in \mathbb{F}^{n \times n} \) invertible such that \( M = B^{-1}NB \). Show that \( \text{rank}(M) = \text{rank}(N) \).

### 38

Let \( B = \{1 + x, x^3, x^4 + x^8\} \subset \mathcal{P}_9 \) and \( V = \text{Span}(B) \). Then \( B \) is a basis of \( V \) (you do not need to show this). Suppose \( L: V \to V \) is given by

\[
[ L ]_B = A = \begin{pmatrix} 2 & 9 & -9 \\ 0 & 14 & -12 \\ 0 & 9 & -7 \end{pmatrix}.
\]

a) What is \( L(1 + x - x^3) \)?

b) Determine the eigenvalues of \( A \).

c) Determine a matrix \( S \) such that \( S^{-1}AS \) is diagonal.

d) Determine a basis \( \mathcal{C} \) of \( V \) such that \( [ L ]_\mathcal{C} \) is diagonal.

Problems marked with a * are bonus problems for extra credit.