

12) Which of the following sets are spanning sets for \mathbb{R}^3 ? In each case show that the set spans if true; respectively give an example of a vector in \mathbb{R}^3 that does not lie in the span if false.

- a) $\{(1, 2, 0)^T, (0, 1, 0)^T, (0, 2, 1)^T\}$
- b) $\{(1, 2, 0)^T, (0, 1, 0)^T, (0, 2, 1)^T, (1, 2, 3)^T\}$
- c) $\{(2, 1, -2)^T, (3, 2, -2)^T, (2, 2, 0)^T, (4, 3, -2)\}$
- d) $\{(2, 1, -2)^T, (3, 2, -2)^T, (2, 2, 0)^T\}$
- e) $\{(1, 1, 1)^T\}$

13) Which of the following sets are spanning sets for \mathcal{P}_3 (the set of polynomials of degree smaller than 3)? Show that the set spans, or give an example of a polynomial that does not lie in the span.

- a) $\{x - 1, x^2 - 1, 3x^2 + 2x - 5\}$ (Hint: What is $f(1)$ for the elements in the span?)
- b) $\{x^2 + 1, 2x + 3, x\}$
- c) $\{1 + x, x + x^2, 1 + x^2, 1 - x^2\}$

14) Which of the sets in problem 12 and 13 are linearly independent? (Prove that they are independent, or show a nontrivial linear relation among the vectors.)

15) Let $A = \begin{pmatrix} 8 & -1 & 0 & 23 \\ -16 & 2 & 1 & -44 \\ 9 & -1 & 0 & 26 \end{pmatrix}$. Determine a spanning set for the nullspace of A .

16) Show: Let V be a vector space, $X \subset V$ and $\underline{y} \in V$. We form a new set $Z = X \cup \{\underline{y}\}$ by adding the vector \underline{y} to X .

- a) Show that if $\underline{y} \in \text{Span}(X)$ then Z is linearly dependent.
- b) Assume that X is linearly independent. Show that if Z is linearly dependent, then $\underline{y} \in \text{Span}(X)$.

17*) a) Let V be a vector space and $S, T \leq V$ be two subspaces. Show that the intersection $S \cap T$ is a subspace of V .

b) Give an example of a vector space V and subspaces $S, T \leq V$ such that the union $S \cup T$ is not a subspace of V .

Spanning and Linear independence in function spaces

Some books deal with this topic defining a “Wronski Matrix”. This is rather complicated and confusing, requires unnecessary extra work, and only works if the functions are differentiable. Don’t bother with it!

When we show linear dependence or spanning for column vectors, we want to express a column vector (the zero vector in case of linear independence or the column which we want to test to be contained in the span) as a linear combination of given columns. This is done by solving a system of linear equations.

Suppose we would work with 3 columns with 1000 entries. Then we have a system with 3 variables and 1000 equations. Instead of solving this huge system, we can start by considering only the first few equations and solve this partial system. If this smaller system has

no solution then the large system has no solution, and we can stop.

only the zero solution if we are testing for linear independence then this is the only solution (it will automatically fulfill the other equations, as a homogeneous system has always solutions) and we can stop.

exactly one solution in general. Then we can verify that this solution also fulfills the other equations (this is substantially less work).

More than one solution Then we need to consider more equations, but probably still not all 1000.

If we have a function space, we can consider functions as “infinitely long” vectors that give the values at all the points in the domain. We can therefore use the same approach. Picking equations from some entries now corresponds to evaluating the functions at only a few points.

This gives the following approach for testing linear independence of n functions f_1, \dots, f_n ; respectively testing that g is in $\text{Span}(f_1, \dots, f_n)$:

1. Pick n “random points a_1, \dots, a_n . (In practice you will pick points that are also convenient, say $0, 1, 2, \dots$)
2. Evaluate the functions at the points. This gives you column vectors

$$\begin{pmatrix} f_1(a_1) \\ \vdots \\ f_1(a_n) \end{pmatrix}, \dots, \begin{pmatrix} f_n(a_1) \\ \vdots \\ f_n(a_n) \end{pmatrix}, \begin{pmatrix} g(a_1) \\ \vdots \\ g(a_n) \end{pmatrix},$$

(When testing linear independence g is the zero function and thus corresponds to a zero column.)

3. Now solve the system of equations given by these columns (this corresponds to solving only a few of the equations in the example of long vectors).
4. If you get

no solution the system has no solutions.

only the zero solution and you are testing for linear independence, then the functions are independent

exactly one solution then test (using algebra and relations among the functions) whether the coefficients you found work in fact for every argument value. If they do you have found an expression. If not, you find another point a_{n+1} at which the only possible relation does not hold, and thus there is no solution.

multiple solutions In this case our points are badly chosen. Just chose different points (or an extra point) and go back to step 2.