

67) Suppose we are making pairs (a, b) of measurements in some experiment and end up with the following six pairs:

19	22	6	3	2	20
12	6	9	15	13	5

- a) Shift the data such that the average of the pairs becomes $(0,0)$
- b) Determine the correlation matrix $C \in \mathbb{R}^{2 \times 2}$.
- c) Find an orthogonal matrix U such that $U^{-1}CU$ is diagonal.
- d) Determine principal components (i.e. new coordinates in which the correlation matrix is diagonal) for this data.

68) Let $A = U\Sigma V^T$ be a singular value decomposition. Show that the rank of A is equal to the number of nonzero singular values in Σ .

69*) Suppose the singular values of an invertible matrix $A \in \mathbb{R}^{n \times n}$ are $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n$.

a) Show that for every $\underline{v} \in \mathbb{R}^n$, we have that $\|A\underline{v}\| \leq \sigma_1 \|\underline{v}\|$ (where $\|\cdot\|$ is the standard inner product on \mathbb{R}^n), with equality attained for some vector $\underline{v} \neq \underline{0}$.

b) Show similarly, that for $\underline{v} \in \mathbb{R}^n$, we have that $\|A^{-1}\underline{v}\| \leq \frac{1}{\sigma_n} \|\underline{v}\|$.

Note: This indicates that the number $\kappa = \sigma_1/\sigma_n$ indicates how much vectors can change under multiplication by A . κ is called the condition number of A . A large condition number indicates that a matrix is close to being singular, and that small errors on a vector \underline{b} can have a large impact on the solutions of $A\underline{x} = \underline{b}$.

70*) Prove that if A and B are similar matrices, then

- a) A and B have the same eigenvalues with the same arithmetic and the same geometric multiplicities.
- b) For each eigenvalue λ and each i we have that $\dim N(A - \lambda \cdot I)^i = \dim N(B - \lambda \cdot I)^i$.

71) Write down the Jordan Canonical form of a matrix A which has the following kernel dimensions:

i	1	2	3	4	5	6
$\dim N(A - 5 \cdot I)^i$	6	11	14	17	18	18
$\dim N(A + 2 \cdot I)^i$	2	4	5	5	5	5

72) Why is the following alleged sequence of kernel dimensions for the eigenvalue λ of a linear transformation L impossible?

i	0	1	2	3	4
$\dim(\ker((L - \lambda \text{id})^i))$	0	1	2	6	6

73*) How many different matrices in Jordan Canonical Form exist that have the characteristic polynomial $(x - 1)^5(x - 2)^2$ are there? Write them all down.

Problems marked with a * are bonus problems for extra credit.