51) We give four points \( P = (1,1,0), \ Q = (0,1,1), \ R = (1,2,0) \) and \( S = (4, -3, 2) \) in a 3-dimensional space by their coordinates. Let \( E \) be the plane that goes through the points \( P, \ Q \) and \( R \). We want to determine the distance of \( S \) to this plane \( E \):

a) Perform a shift such that point \( P \) becomes the origin \( P' = (0,0,0)^T \) and \( Q, \ R, \ S \) change accordingly to \( Q' = (-1,0,1), \ R' = (0,1,0), \ S' = (3,-4,2) \). Then the plane \( E' \) through \( P', \ Q', \ R' \) goes through the origin and corresponds to a subspace of \( \mathbb{R}^3 \). Determine a basis \( B \) of this (2-dimensional) subspace \( U \).

b) Determine a basis \( C \) for \( U^\perp \). Now \( B \) and \( C \) together form a basis of \( \mathbb{R}^3 \).

c) Express the vector to the point \( S' \) as a linear combination of the vectors from \( B \) and \( C \).

d) Find the point on \( E' \) that is closest to \( S' \). (Hint: What does the linear combination in c) tell us?)

e) What is the distance between \( E \) and \( S \)?

52) In \( V = C[0,2] \) with inner product \( \langle f, g \rangle = \int_0^2 f(x)g(x) \, dx \), let \( U = \mathbb{P}_2 \) (i.e. polynomials of degree \( \leq 2 \)). Calculate the orthogonal projection \( P_U(\exp(x)) \) of \( \exp(x) \) onto \( U \).

(Hint: use partial integration to evaluate \( \int_0^2 x^j e^x \, dx \).)

53) Determine the least-squares quadratic polynomial \( y = c_0 + c_1 t + c_2 t^2 \) for approximating the following data set \( \{(t,y)\} \):
\[ \{(0,5), (1,3),(2,4), (3,8), (4,12), (5,20)\} \]

54) We want to find out the value of three variables \( x_1, x_2, x_3 \). From measurements we get the equations
\[
\begin{align*}
2x_1 + 4x_2 + 1x_3 &= 1 \\
1x_1 + 3x_2 + 4x_3 &= 2 \\
3x_1 + 5x_3 &= 1
\end{align*}
\]

a) Does this system have a solution? What is the solution?

b) To reduce influence of measurement errors, we make some more measurements and get three further equations:
\[
\begin{align*}
5x_1 - 1x_2 &= 0 \\
7x_1 + 6x_2 + 10x_3 &= 1 \\
-3x_1 - 4x_3 &= 1
\end{align*}
\]

Does the system of the six equations (form a) and b) together) have a solution?

c) What values for \( x_1, x_2, x_3 \) should we deduce from these six equations?
Let $V$ be an inner product space and $S \subseteq V$. Let $\mathcal{B}$ be a basis for $S$ and $\mathcal{C}$ be a basis for $S^\perp$.

a) Show that the union $\mathcal{B} \cup \mathcal{C}$ is linearly independent. (Hint: Consider the orthogonal projection onto $S$.)

b) Show that $\mathcal{B} \cup \mathcal{C}$ is a basis for $V$.

Problems marked with an * are bonus problems for extra credit.

Note the changed due date! Monday April 3, as April 5 is Midterm 2.

Please note that Dr. Painter will not conduct his review sessions the week of March 27-31.

Sample Midterm Problems

We will have the second Midterm on Wednesday, April 5. Setup is as before: The midterm will cover the material up to this homework sheet. There will be 5 problems in total. You are permitted to bring a calculator (no PDA, laptop or similar).

The following problems are (in addition to all the homework problems on the previous sheets) typical midterm problems. (This does not mean that every midterm problem will be exactly of a type as given here.)

1) Let $V = P_3$ with basis $\mathcal{B} = (1, x, x^2)$ and let $L: V \to \mathbb{R}^2$, $L(f) \mapsto (f(1), f(2))^T$.

a) Show that $L$ is linear.

b) For the standard basis $S$ of $\mathbb{R}^2$, calculate $S[\mathcal{L}]_{\mathcal{B}}$.

c) Determine the dimension of $\ker(L)$.

2) Let $\mathcal{B} = \{b_1 = (0, 1, 4)^T, b_2 = (1, 2, 3)^T\} \subseteq \mathbb{R}^3$ and $V = \text{Span}(\mathcal{B})$. Let $\mathcal{C} = \{c_1 = (3, 7, 13)^T, c_2 = (2, 5, 10)^T\}$. (You may assume that $\mathcal{B}$ and $\mathcal{C}$ both are linearly independent.)

a) Show that $\text{Span}(\mathcal{C}) = V$. Explain your reasoning.

b) Determine $\mathcal{C}[\text{id}]_{\mathcal{C}}$.

c) Let $L: V \to V$ be a linear transformation such that $L(c_1) = c_1 + c_2$ and $L(c_2) = c_1 - c_2$. Determine $\mathcal{C}[L]_{\mathcal{C}}$ and $\mathcal{B}[L]_{\mathcal{B}}$.

3) Which of the following matrices can be diagonalized? Justify your answer. (You do not need to perform the actual diagonalization!)

\[
A = \begin{pmatrix} 8 & -6 & -9 \\ 2 & -1 & -2 \\ 2 & -2 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

\[
\chi_A(x) = (x-1)(x-2)(x-3), \quad \chi_B(x) = (x-2)^2(x-3), \quad \chi_C(x) = (x-2)^2(x-3), \quad \chi_D(x) = (x-1)^3
\]
4) Let $A, B \in \mathbb{F}^{n \times n}$ such that $AB = BA$. Let $\lambda$ be an eigenvalue of $A$. Show that the corresponding eigenspace of $A$ (this space is $N(A - \lambda I)$) is invariant under $B$.

5) Let $A = \begin{pmatrix} 4 & 0 \\ 3 & 1 \end{pmatrix}$.
   a) Show that $A$ can be diagonalized and find a matrix $S$ such that $S^{-1}AS$ is diagonal.
   b) Find two different matrices $R_1$ and $R_2$ such that $R_1^2 = R_2^2 = A$.
   (Hint: Consider first the case of a diagonal matrix.)

6) Determine the least-squares approximation by a linear function (i.e. a polynomial of the form $a + bx$ for the following data set:

   $(0, 35), (1, 20), (2, 15), (3, 10)$

7) a) Let $V$ be a vector space with inner product $\langle \cdot, \cdot \rangle$ and $w \in V$ be a particular vector. Show: The map $L: V \rightarrow \mathbb{R}$ defined by setting $L(v) = \langle w, v \rangle$ is linear.
   b) Let $V = \text{Span}(1, x, x^2)$ with inner product $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$ and $w = 1 + x \in V$. Determine $\langle 1, w \rangle$, $\langle x, w \rangle$ and $\langle x^2, w \rangle$.
   c) For $V$ and $w$ as defined in b) define $L: V \rightarrow \mathbb{R}$ by $L(v) = \langle w, v \rangle$. (By part a) $L$ is linear.) Determine $S[L]_B$, where $S = (1)$ is the standard basis of $\mathbb{R}$ and $B = (1, x, x^2)$.

8) Let $V = C[0, 1]$ with the inner product $\langle f, g \rangle = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x)g(x)dx$. Let $B = (1, x^2)$ and $U = \text{Span}(B) \leq V$.
   a) Compute the Gram matrix $G_B$ for $B$.
   b) Compute the orthogonal projection of $\sin(x)$ onto $U$.
   (Hint: $\int x^2 \cdot \cos(x) \, dx = (x^2 - 2) \sin(x) + 2x \cos(x).$)

9) Consider the following matrix over the complex numbers:

   \[
   A = \begin{pmatrix}
   6 & -6 & 3 & 14 \\
   41 & -107 & 20 & 107 \\
   -2 & 8 & -1 & -3 \\
   39 & -106 & 19 & 102 \\
   \end{pmatrix}, \quad \chi_A(x) = (x - 1)(x + 1)(x + i)(x - i),
   \]

   where $i = \sqrt{-1}$.

   a) What are the eigenvalues of $A$? Show that $A$ is diagonalizable and determine a diagonal matrix $D$ such that $D = S^{-1}AS$ for some suitable matrix $S$ (you do not need to determine $S$)?
   b) Determine $D^{2001}$ (Note that $2001 = 4 \cdot 500 + 1$).
   c) Determine $A^{2001}$. (Hint: You do not need to compute eigenvectors explicitly.) Explain your reasoning!