Mathematics 369 **Homework (due Apr 3)** A. Hulpke

51) We give four points $P = (1,1,0)$, $Q = (0,1,1)$, $R = (1,2,0)$ and $S = (4,-3,2)$ in a 3-dimensional space by their coordinates. Let E be the plane that goes through the points P, Q and R. We want to determine the distance of S to this plane E:
a) Perform a shift such that point P becomes the origin $P' = (0,0,0)^T$ and Q, R, S change accordingly to $Q' = (-1,0,1)$, $R' = (0,1,0)$, $S' = (3,-4,2)$. Then the plane E' through P', Q', R' goes through the origin and corresponds to a subspace of $\mathbb{R}^3$. Determine a basis $\mathcal{B}$ of this (2-dimensional) subspace U.
b) Determine a basis $\mathcal{C}$ for $U^\perp$. Now $\mathcal{B}$ and $\mathcal{C}$ together form a basis of $\mathbb{R}^3$.
c) Express the vector to the point S' as a linear combination of the vectors from $\mathcal{B}$ and $\mathcal{C}$.
d) Find the point on E' that is closest to S'. (Hint: What does the linear combination in c) tell us?)
e) What is the distance between E and S?

52) In $V = C[0,2]$ with inner product $\langle f,g\rangle = \int_0^2 f(x)g(x)\,\mathrm{d}x$, let $U = \mathcal{P}_2$ (i.e. polynomials of degree ≤ 2). Calculate the orthogonal projection $P_U(\exp(x))$ of $\exp(x)$ onto U.
(**Hint:** use partial integration to evaluate $\int_0^2 x^j e^x\,\mathrm{d}x$.)

53) Determine the least-squares quadratic polynomial $y = c_0 + c_1 t + c_2 t^2$ for approximating the following data set $\{(t,y)\}$:
$\{(0,5),(1,3)(2,4),(3,8),(4,12),(5,20)\}$

54) We want to find out the value of three variables x_1,x_2,x_3. From measurements we get the equations

$$\begin{array}{llll} 2x_1 & +4x_2 & +1x_3 & =1 \\ 1x_1 & +3x_2 & +4x_3 & =2 \\ 3x_1 & & +5x_3 & =1 \end{array}$$

a) Does this system have a solution? What is the solution?
b) To reduce influence of measurement errors, we make some more measurements and get three further equations:

$$\begin{array}{llll} 5x_1 & -1x_2 & & =0 \\ 7x_1 & +6x_2 & +10x_3 & =1 \\ -3x_1 & & -4x_3 & =1 \end{array}$$

Does the system of the six equations (form a) and b) together) have a solution?
c) What values for x_1,x_2,x_3 should we deduce from these six equations?

55*) Let V be a n inner product space and $S \leq V$. Let $\mathcal{B}$ be a basis for S and $\mathcal{C}$ be a basis for $S^{\perp}$.
a) Show that the union $\mathcal{B} \cup \mathcal{C}$ is linearly independent. (**Hint:** Consider the orthogonal projection onto S.)
b) Show that $\mathcal{B} \cup \mathcal{C}$ is a basis for V.
Problems marked with a *are bonus problems for extra credit.

Note the changed due date! Monday April 3, as April 5 is Midterm 2.
Please note that Dr. Painter will not conduct his review sessions the week of March 27-31.

Sample Midterm Problems

We will have the second Midterm on Wednesday, April 5. Setup is as before: The midterm will cover the material up to this homework sheet. There will be 5 problems in total. You are permitted to bring a calculator (no PDA, laptop or similar).
The following problems are (in addition to all the homework problems on the previous sheets) typical midterm problems. (This does **not** mean that every midterm problem will be exactly of a type as given here.)

1) Let $V = \mathcal{P}_3$ with basis $\mathcal{B} = (1, x, x^2)$ and let $L: V \to \mathbb{R}^2$, $L: f(x) \mapsto (f(1), f(2))^T$.
a) Show that L is linear.
b) For the standard basis $\mathcal{S}$ of $\mathbb{R}^2$, calculate ${}_{\mathcal{S}}[L]_{\mathcal{B}}$.
c) Determine the dimension of $\ker(L)$.

2) Let $\mathcal{B} = \{\underline{\mathbf{b_1}} = (0,1,4)^T, \underline{\mathbf{b_2}} = (1,2,3)^T\} \subset \mathbb{R}^3$ and $V = \mathrm{Span}(\mathcal{B})$. Let $\mathcal{C} = \{\underline{\mathbf{c_1}} = (3,7,13)^T, \underline{\mathbf{c_2}} = (2,5,10)^T\}$. (You may assume that $\mathcal{B}$ and $\mathcal{C}$ both are linearly independent.)
a) Show that $\mathrm{Span}(\mathcal{C}) = V$. Explain your reasoning.
b) Determine ${}_{\mathcal{B}}[\mathrm{id}]_{\mathcal{C}}$.
c) Let $L: V \to V$ be a linear transformation such that $L(\underline{\mathbf{c_1}}) = \underline{\mathbf{c_1}} + \underline{\mathbf{c_2}}$ and $L(\underline{\mathbf{c_2}}) = \underline{\mathbf{c_1}} - \underline{\mathbf{c_2}}$. Determine ${}_{\mathcal{C}}[L]_{\mathcal{C}}$ and ${}_{\mathcal{B}}[L]_{\mathcal{B}}$.

3) Which of the following matrices can be diagonalized? Justify your answer. (You do **not** need to perform the actual diagonalization!)

$$A = \begin{pmatrix} 8 & -6 & -9 \\ 2 & -1 & -2 \\ 2 & -2 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad C = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\chi_A(x) = (x-1)(x-2)(x-3) \quad \chi_B(x) = (x-2)^2(x-3) \quad \chi_C(x) = (x-2)^2(x-3) \quad \chi_D(x) = (x-1)^3$$

4) Let $A, B \in \mathbb{F}^{n\times n}$ such that $AB = BA$. Let λ be an eigenvalue of A. Show that the corresponding eigenspace of A (this space is $N(A-\lambda I)$) is invariant under B.

5) Let $A = \begin{pmatrix} 4 & 0 \\ 3 & 1 \end{pmatrix}$.
a) Show that A can be diagonalized and find a matrix S such that $S^{-1}AS$ is diagonal.
b) Find two different matrices R_1 and R_2 such that $R_1^2 = R_2^2 = A$.
(**Hint:** Consider first the case of a diagonal matrix.)

6) Determine the least-squares approximation by a linear function (i.e. a polynomial of the form $a+bx$ for the following data set:

$$(0,35),(1,20),(2,15),(3,10)$$

7) a) Let V be a vector space with inner product $\langle\cdot,\cdot\rangle$ and $\underline{\mathbf{w}} \in V$ a particular vector. Show: The map $L: V \to \mathbb{R}$ defined by setting $L(\underline{\mathbf{v}}) = \langle \underline{\mathbf{w}}, \underline{\mathbf{v}} \rangle$ is linear.
b) Let $V = \text{Span}(1, x, x^2)$ with inner product $\langle \underline{\mathbf{f}}, \underline{\mathbf{g}} \rangle = \int_0^1 f(x)g(x)\,\mathrm{d}x$ and $\underline{\mathbf{w}} = 1 + x \in V$. Determine $\langle 1, \underline{\mathbf{w}} \rangle$, $\langle x, \underline{\mathbf{w}} \rangle$ and $\langle x^2, \underline{\mathbf{w}} \rangle$.
c) For V and $\underline{\mathbf{w}}$ as defined in b) define $L: V \to \mathbb{R}$ by $L(\underline{\mathbf{v}}) = \langle \underline{\mathbf{w}}, \underline{\mathbf{v}} \rangle$. (By part a) L is linear.) Determine ${}_{\mathcal{S}}[L]_{\mathcal{B}}$, where $\mathcal{S} = (1)$ is the standard basis of $\mathbb{R}$ and $\mathcal{B} = (1, x, x^2)$.

8) Let $V = C[0,1]$ with the inner product $\langle f, g \rangle = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x)g(x)\,\mathrm{d}x$. Let $\mathcal{B} = (1, x^2)$ and $U = \text{Span}(\mathcal{B}) \leq V$.
a) Compute the Gram matrix $G_{\mathcal{B}}$ for $\mathcal{B}$.
b) Compute the orthogonal projection of $\sin(x)$ onto U.
Hint: $\int x^2 \cdot \cos(x)\,\mathrm{d}x = (x^2-2)\sin(x) + 2x\cos(x)$.)

9) Consider the following matrix over the complex numbers:

$$A = \begin{pmatrix} 6 & -6 & 3 & 14 \\ 41 & -107 & 20 & 107 \\ -2 & 8 & -1 & -3 \\ 39 & -106 & 19 & 102 \end{pmatrix}, \qquad \chi_A(x) = (x-1)(x+1)(x+i)(x-i),$$

where $i = \sqrt{-1}$.
a) What are the eigenvalues of A? Show that A is diagonalizable and determine a diagonal matrix D such that $D = S^{-1}AS$ for some suitable matrix S (you do **not** need to determine S)?
b) Determine D^{2001} (Note that $2001 = 4 \cdot 500 + 1$).
c) Determine A^{2001}. (Hint: You do **not** need to compute eigenvectors explicitly.) Explain your reasoning!