Adjoint and Orthogonal Operators

We will consider several properties of operators when interplaying with an inner product. These properties are usually defined abstractly for an operator and an inner product. Once we choose a basis (and thus can represent operators by matrices) we can translate the properties to properties of matrices (which are sometimes called slightly different). The properties for operators also work (with the same definitions but much harder proofs) for infinite-dimensional spaces.

In the following we will assume for the purposes of matrices that we are working with an orthonormal basis $\mathcal{B}$ (for example the standard basis of $\mathbb{F}^n$). With respect to this basis the inner product is the standard inner product with trivial Gram matrix.

We will give some formulations for the case of $\mathbb{F} = \mathbb{C}$. If $\mathbb{F} = \mathbb{R}$ simply ignore all complex conjugation.

We will assume that V is an inner product space with (orthonormal) basis $\mathcal{B}$ (so the inner product is $\langle \underline{\mathbf{v}}, \underline{\mathbf{w}} \rangle = [\underline{\mathbf{v}}]_{\mathcal{B}}^T \cdot \overline{[\underline{\mathbf{w}}]_{\mathcal{B}}}$) and that $L: V \to V$ is linear. We also let $A = {}_{\mathcal{B}}[L]_{\mathcal{B}}$.

Name	Definition for Operators	Name for Matrices	Definition for matrices
adjoint	The **adjoint** of L is the map $L^*: V \to V$ defined by $\langle \underline{\mathbf{v}}, L(\underline{\mathbf{w}}) \rangle = \langle L^*(\underline{\mathbf{v}}), \underline{\mathbf{w}} \rangle$ for all $\underline{\mathbf{v}}, \underline{\mathbf{w}} \in V$. One can show that L^*i, defined this way, is unique and linear.	conjugate transpose	$[\underline{\mathbf{v}}]_{\mathcal{B}}^T \cdot \overline{A \cdot [\underline{\mathbf{w}}]_{\mathcal{B}}} = \overline{A^T} \cdot [\underline{\mathbf{v}}]_{\mathcal{B}}^T \cdot \overline{[\underline{\mathbf{w}}]_{\mathcal{B}}}$. Write: $A^* = \overline{A^T}$.
self-adjoint	L is self-adjoint if $L = L^*$	symmetric ($\mathbb{R}$), hermitian ($\mathbb{C}$)	$A = \overline{A^T} = A^*$
orthogonal ($\mathbb{R}$), unitary ($\mathbb{C}$)	L is orthogonal (unitary) if $\langle L(\underline{\mathbf{v}}), L(\underline{\mathbf{w}}) \rangle = \langle \underline{\mathbf{v}}, \underline{\mathbf{w}} \rangle$ for all $\underline{\mathbf{v}}, \underline{\mathbf{w}} \in V$. In particular this implies that $\|\underline{\mathbf{v}}\} = \|L(\underline{\mathbf{v}})\|$ for all $\underline{\mathbf{v}} \in V$.	orthogonal ($\mathbb{R}$), unitary ($\mathbb{C}$)	$I = A \cdot \overline{A^T} = AA^*$, thus $A^* = A^{-1}$. Respectively: The columns (or: the rows) of A form an orthonormal basis with respect to the standard inner product on $\mathbb{F}^n$.
normal	L is normal if $\|L(\underline{\mathbf{v}})\| = \|L^*(\underline{\mathbf{v}})\|$ for all $\underline{\mathbf{v}} \in V$. This is equivalent to the property that $L(L^*(\underline{\mathbf{v}})) = L(L^*(\underline{\mathbf{v}}))$ for all $\underline{\mathbf{v}} \in V$.	normal	$AA^* = A^*A$

Note that self-adjoint operators are normal. The **spectral theorem** in the most general form says that L has an orthonormal basis of eigenvectors if and only if L is normal. This means:

over $\mathbb{C}$ If A is normal (or hermitian) there is a unitary matrix U such that $U^{-1}AU = U^*AU$ is diagonal.

over $\mathbb{R}$ If A is symmetric there is an orthogonal matrix O such that $O^{-1}AO = O^TAO$ is diagonal. I.e. all eigenvalues of A are real and there is an orthonormal basis consisting of eigenvectors of A.