You can work on the problems in any order you like. Show your work! All problems carry the same weight.
Calculation steps and explanation for statements made are a crucial part of a solution. Partial credit will be given sparingly – rather complete one problem than start two only partially.

1) Find an integer solution \( x \) for the equation

\[
23x \mod 89 = 4
\]

\[
\text{gcd}(23, 89) : \quad 89 = 3 \cdot 23 + 20 \implies 20 = 89 - 3 \cdot 23
\]

\[
23 = 1 \cdot 20 + 3 \quad \implies 3 = 23 - 20
\]

\[
20 = 6 \cdot 3 + 2 \quad \implies 2 = 20 - 6 \cdot 3
\]

\[
3 = 1 \cdot 2 + 1 \implies 1 = 3 - 2
\]

\[
\implies 1 = 3 - 2 = 3 - 20 + 6 \cdot 3 = 7 \cdot (23 - 20) - 20 = 7 \cdot 23 - 8(89 - 3 \cdot 23)
\]

\[
= -8(89) + 7 \cdot 23 \implies 23 \equiv 23 \pmod{89}
\]

\[
\implies x \equiv 4 \cdot 31 \equiv 124 \equiv 35 \pmod{89}
\]
2) Determine (with explanation) the units and zero divisors in the ring $\mathbb{Z}_{10}$.

As $\gcd(10, 1) = \gcd(10, 3) = \gcd(10, 7) = \gcd(10, 9) = 1$

$1, 3, 7, 9$ are units.

As $2 \cdot 5 = 4 \cdot 5 = 6 \cdot 5 = 8 \cdot 5 = 0 \pmod{10}$

$2, 4, 5, 6, 8, 9$ are zero divisors.

As units cannot be zero divisors and we exhausted $\mathbb{Z}_{10} \setminus \{0\}$, these are all.
3) Construct the multiplication table for the field with 4 elements by using polynomial arithmetic modulo 2 and modulo $x^2 + x + 1$.

In this field, calculate $\frac{a}{a+1}$, where $a$ is the element represented by the polynomial $x$.

$$\text{Elements} = \{0, 1, x, x+1\}$$

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To calculate $\frac{a}{a+1}$, note that $x - (x+1) = 1 \Rightarrow \frac{1}{x+1} = x$.

$$\Rightarrow \frac{a}{a+1} = \frac{x}{x+1} = x \cdot \frac{1}{x+1} = x \cdot x = x^2 \equiv x+1 \pmod{x^2 + x + 1}$$

$$\Rightarrow \frac{a}{a+1} = a+1 = x+1$$
4) Let $R = \mathbb{Q}[x]$ and let $A = \{ f \in R \mid f(1) = 0 \}$.

a) Show that $A$ is a subring of $R$.

b) Show that $A$ is an ideal in $R$.

c) Let $B = \{ f \in R \mid f(1) = 1 \}$. Show that $B$ is not a subring of $R$.

\begin{itemize}
  \item[a)] Let $A$. Clearly $\hat{d}(x) = 0$ is an element of $A = A \neq \emptyset$.

  \textbf{Let $f, g \in A$.}

  \textbf{Then} $\hat{(f+g)}(1) = \hat{f}(1) + \hat{g}(1) = 0 + 0 = 0 \Rightarrow f + g \in A$.

  \textbf{Similarly} $\hat{(-f)}(1) = -f(1) = -0 = 0 \Rightarrow -f \in A$.

  \textbf{Also} $\hat{(fg)}(1) = f(1)g(1) = 0 \cdot 0 = 0$.

  \textbf{Thus} $A \leq R$.

  \item[b)] Let $d \in A$, $g \in R$. Then $\hat{(dg)}(1) = \hat{d}(1)g(1)$.

  \textbf{Sufficient, as $\hat{d}$ is constant.}

  \textbf{Thus} $d \cdot g \in A$.

  \textbf{Thus} $(a, A \leq R) \Rightarrow A \leq R$.

  \item[c)] Let $d \in A$, $\hat{d}(x) = x$. Then $\hat{f}(x) \in B$.

  \textbf{But} $\hat{(f+g)}(1) = f(1) + g(1) = 1 + 1 = 2 \neq 1$.

  \textbf{Thus} $f + 1 \notin B$, and therefore $B \neq R$.
5) Let $R$ be a commutative ring with one, in which $R$ and $(0)$ are the only ideals. For $0 \neq x \in R$ consider the ideal generated by $x$, which is $A = (x) = \{ax \mid a \in R\}$. Show that $A \neq (0)$ and thus $A = R$. Conclude that $x$ must have an inverse and therefore conclude that $R$ is a field.

Your proof should consist of full sentences. (It is not sufficient to simply point to lecture or homework.)