23) All pw: 36^4 poss.

a) No digit: 26^4. So $36^4 - 26^4$ at least one

b) Exactly one: $\binom{4}{1} \cdot 10 \cdot 26^3$

position dig. 3 letters left.

c) No digits: 26^4 +

1 digit: $\binom{4}{1} \cdot 10 \cdot 26^3$ +

2 digits: $\binom{4}{2} \cdot 10^2 \cdot 26^2$

24) Use Frequencies per position (as the k-th letter in each text is encrypted the same.

26) a) The only numbers that do not have gcd $= 1$ are $p, 2p, 3p, \ldots, (p-1)\cdot p, p^2$. These are p numbers out of p^2, so there are $p^2 - p = p(p-1)$ numbers with gcd $= 1$

b) Now the factors of and tuples of p are

$1\,p, 2p, \ldots (p-1)p, p\cdot p, (p+1)p, \ldots, p^k - p = (p^{k-1}-1)p, p^k$

That is, the factors are $1, \dots, p^{k-1}$

So p^{k-1} numbers have $\gcd \neq 1$, so $p^k - p^{k-1}$

$= p^{k-1}(p-1)$ numbers have $\gcd = 1$.

c/d) Consider the following diagram for the numbers $1 \dots n$:

$$
v\left\{
\begin{array}{cccc}
\multicolumn{4}{c}{\overbrace{\hspace{8cm}}^{u}} \\
1 & 2 & 3 \dots & u \\
u+1 & u+2 & \dots & 2u \\
2u+1 & 2u+2 & \dots & 3u \\
\vdots & & & \\
 & & & (v-1)u \\
(v-1)u+1 & \dots & \dots & vu = n
\end{array}
\right.
$$

Then $\gcd(a,u) \neq 1$ if and

only if $\gcd(a \bmod u, u) \neq 1$

Thus there are $\varphi(u)$ columns of numbers (namely those whose top entry has gcd $\neq 1$) which have gcd $\neq 1$ with u.

The same happens (transposed diagram) with row indices and divisors of v. As $\gcd(u,v)=1$ there is no interference. Thus a number has gcd $=1$ with u,v if neither row index, nor column index has nontriv. gcd.

27] $\varphi(2^4 \cdot 3^2 \cdot 5 \cdot 11) = \varphi(2^4)\cdot\varphi(3^2)\cdot\varphi(5)\cdot\varphi(11)$
$= 2^3 \cdot 2 \cdot 3 \cdot 4 \cdot 10$

28] a) If the glass survives a ft and breaks at $a+2$ feet (having skipped $a+1$ ft) we can never distinguish $a+1$ and $a+2$ ft. as max incl. height.
b) Use glass 1 at 10, 20, 30, ... ft. If it breaks at a ft, use glass 2 at $a-9, a-8, \ldots, a-1$ ft.
This is best possible, as a square has the shortest circumference of all rectangles with same area