Mathematics 301 **Homework (due Oct. 25 – two weeks)** A. Hulpke

30) Let $\Gamma = (V, E)$ be a graph. We define the *complement graph* as

$$\bar{\Gamma} = (V, \{\{a, b\} \mid \{a, b\} \notin E\}),$$

that is we swap the status of edges and non-edges. Show that if Γ is not connected that $\bar{\Gamma}$ is connected.

32) Let G be a graph on n vertices with adjacency matrix A. Let I_n be the $n \times n$ identity matrix. Show that G is connected if and only if the matrix $(I_n + A)^{n-1}$ has no zeroes.

33) Show that a graph with 2 or more vertices always has two vertices of the same degree.

34) Show that a graph isomorphism must preserve the distance of vertices, that is if f is an isomorphism and $x, y \in V$ then $d(x, y) = d(f(x), f(y))$.

35) Show that the distance function $d(x, y)$ for vertices satisfies the following conditions:

1. $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$.
2. (Symmetry) $d(x, y) = d(y, x)$.
3. (Triangle Inequality) $d(x, z) \leq d(x, y) + d(y, z)$ for any three vertices x, y, z. (**Hint:** Combine the paths to a walk and use the result from class.)

(Thus the distace defines a *metric* on the vertices.)

36) Give an example of a graph in which the maximal distance between vertices is 2, but which contains an induced subgraph that is isomorphic to C_7 (a 7-cycle)

37) Give an example of a sequence of length $\geq$ 4 that cannot be the degree sequence of a graph (and explain why this is so).

38) Is there a graph on 5 vertices, whose degree sequence equals $(4, 4, 4, 2, 2)$? Explain!

39 Determine the graphs with degree sequence $(6, 3, 3, 3, 3, 3, 3)$ up to isomorphism. Explain why there cannot be another such graph.

40) Using the Erdős-Gallai theorem, determine which of the following sequences are degree sequences of graphs:

1. $(4, 3, 3, 3, 2, 2, 2, 1)$
2. $(8, 7, 6, 5, 4, 3, 2, 2, 1)$
3. $(5, 5, 5, 3, 3, 3, 3, 3)$
4. $(5, 4, 3, 2, 1, 1, 1, 1, 1, 1, 1, 1)$

B) 1) Show that the isomorphism type of all graphs of degree ≤ 4 is determined uniquely by their degree sequence.
2) Give an example of two graphs (which by a) must be of degree ≥ 5) that are not isomorphic but have the same degree sequence.

Note: The departmental Magnus lectures will be on October 23 and 24 and be on the topic of graph theory. Students who come to the public lecture on October 23, 4pm in TILT 221 (see me at the talk so I record your name) will receive 5 bonus points added to their first midterm score.