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In Exercises 33 38, find the series' interval of convergence and, within this interval, the sum of the series as a function of x.

33. $\sum_{n=0}^{\infty} \frac{(x-1)^{2n}}{4n}$ **34.** $\sum_{n=0}^{\infty} \frac{(x+1)^{2n}}{9^n}$

35. $\sum_{n=0}^{\infty} \left(\frac{\sqrt{x}}{2} - 1\right)^n$ **36.** $\sum_{n=0}^{\infty} (\ln x)^n$

37. $\sum_{n=0}^{\infty} \left(\frac{x^2+1}{3}\right)^n$ **38.** $\sum_{n=0}^{\infty} \left(\frac{x^2-1}{2}\right)^n$

39. For what values of x does the series

$$1 - \frac{1}{2}(x-3) + \frac{1}{4}(x-3)^2 + \cdots + \left(-\frac{1}{2}\right)^n (x-3)^n + \cdots$$

converge? What is its sum? What series do you get if you differentiate the given series term by term? For what values of x does the new series converge? What is its sum?

40. If you integrate the series in Exercise 39 term by term, what new series do you get? For what values of x does the new series converge, and what is another name for its sum?

41. The series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots$$

converges to $\sin x$ for all x.

a. Find the first six terms of a series for $\cos x$. For what values of x should the series converge?

b. By replacing x by $2x$ in the series for $\sin x$, find a series that converges to $\sin 2x$ for all x.

c. Using the result in part (a) and series multiplication, calculate the first six terms of a series for $2 \sin x \cos x$. Compare your answer with the answer in part (b).

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42. The series

34. $\lim_{n \to \infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n \to \infty} \left|\frac{(x+1)^{2n+2}}{9^{n+1}} \cdot \frac{9^n}{(x+1)^{2n}}\right| < 1 \Rightarrow \frac{(x+1)^2}{9} \lim_{n \to \infty} |1| < 1 \Rightarrow (x+1)^2 < 9 \Rightarrow |x+1| < 3$
$\Rightarrow -3 < x+1 < 3 \Rightarrow -4 < x < 2$; when $x = -4$ we have $\sum_{n=0}^{\infty} \frac{(-3)^{2n}}{9^n} = \sum_{n=0}^{\infty} 1$ which diverges; at $x = 2$ we have $\sum_{n=0}^{\infty} \frac{3^{2n}}{9^n} = \sum_{n=0}^{\infty} 1$ which also diverges; the interval of convergence is $-4 < x < 2$; the series $\sum_{n=0}^{\infty} \frac{(x+1)^{2n}}{9^n} = \sum_{n=0}^{\infty} \left(\left(\frac{x+1}{3}\right)^2\right)^n$ is a convergent geometric series when $-4 < x < 2$ and the sum is
$$\frac{1}{1-\left(\frac{x+1}{3}\right)^2} = \frac{1}{\left[\frac{9-(x+1)^2}{9}\right]} = \frac{9}{9-x^2-2x-1} = \frac{9}{8-2x-x^2}$$

39. $\lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{2^{n+1}}\cdot\frac{2^n}{(x-3)^n}\right|<1 \Rightarrow |x-3|<2 \Rightarrow 1<x<5$; when $x=1$ we have $\sum_{n=1}^{\infty}(1)^n$ which diverges; when $x=5$ we have $\sum_{n=1}^{\infty}(-1)^n$ which also diverges; the interval of convergence is $1<x<5$; the sum of this convergent geometric series is $\frac{1}{1+\left(\frac{x-3}{2}\right)}=\frac{2}{x-1}$. If $f(x)=1-\frac{1}{2}(x-3)+\frac{1}{4}(x-3)^2+\ldots+\left(-\frac{1}{2}\right)^n(x-3)^n+\ldots$ $=\frac{2}{x-1}$ then $f'(x)=-\frac{1}{2}+\frac{1}{2}(x-3)+\ldots+\left(-\frac{1}{2}\right)^n n(x-3)^{n-1}+\ldots$ is convergent when $1<x<5$, and diverges when $x=1$ or 5. The sum for $f'(x)$ is $\frac{-2}{(x-1)^2}$, the derivative of $\frac{2}{x-1}$.

41. (a) Differentiate the series for sin x to get $\cos x = 1-\frac{3x^2}{3!}+\frac{5x^4}{5!}-\frac{7x^6}{7!}+\frac{9x^8}{9!}-\frac{11x^{10}}{11!}+\ldots$ $=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\frac{x^{10}}{10!}+\ldots$. The series converges for all values of x since

$$\lim_{n\to\infty}\left|\frac{x^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{x^{2n}}\right|=x^2\lim_{n\to\infty}\left(\frac{1}{(2n+1)(2n+2)}\right)=0<1 \text{ for all } x.$$

(b) $\sin 2x = 2x-\frac{2^3x^3}{3!}+\frac{2^5x^5}{5!}-\frac{2^7x^7}{7!}+\frac{2^9x^9}{9!}-\frac{2^{11}x^{11}}{11!}+\ldots=2x-\frac{8x^3}{3!}+\frac{32x^5}{5!}-\frac{128x^7}{7!}+\frac{512x^9}{9!}-\frac{2048x^{11}}{11!}+\ldots$

(c) $2\sin x\cos x = 2\left[(0\cdot 1)+(0\cdot 0+1\cdot 1)x+\left(0\cdot\frac{-1}{2}+1\cdot 0+0\cdot 1\right)x^2+\left(0\cdot 0-1\cdot\frac{1}{2}+0\cdot 0-1\cdot\frac{1}{3!}\right)x^3\right.$
$+\left(0\cdot\frac{1}{4!}+1\cdot 0-0\cdot\frac{1}{2}-0\cdot\frac{1}{3!}+0\cdot 1\right)x^4+\left(0\cdot 0+1\cdot\frac{1}{4!}+0\cdot 0+\frac{1}{2}\cdot\frac{1}{3!}+0\cdot 0+1\cdot\frac{1}{5!}\right)x^5$
$\left.+\left(0\cdot\frac{1}{6!}+1\cdot 0+0\cdot\frac{1}{4!}+0\cdot\frac{1}{3!}+0\cdot\frac{1}{2}+0\cdot\frac{1}{5!}+0\cdot 1\right)x^6+\ldots\right]=2\left[x-\frac{4x^3}{3!}+\frac{16x^5}{5!}-\ldots\right]$
$=2x-\frac{2^3x^3}{3!}+\frac{2^5x^5}{5!}-\frac{2^7x^7}{7!}+\frac{2^9x^9}{9!}-\frac{2^{11}x^{11}}{11!}+\ldots$