Which of the series in Exercises 1–36 converge, and which diverge? Give reasons for your answers.

1. $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n} + \sqrt[3]{n}}$ 2. $\sum_{n=1}^{\infty} \frac{3}{n + \sqrt{n}}$ 3. $\sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n}$

4. $\sum_{n=1}^{\infty} \frac{1 + \cos n}{n^2}$ 5. $\sum_{n=1}^{\infty} \frac{2n}{3n - 1}$ 6. $\sum_{n=1}^{\infty} \frac{n + 1}{n^2\sqrt{n}}$

7. $\sum_{n=1}^{\infty} \left(\frac{n}{3n + 1}\right)^n$ 8. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3 + 2}}$ 9. $\sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$

10. $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^2}$ 11. $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$ 12. $\sum_{n=1}^{\infty} \frac{(\ln n)^3}{n^3}$

13. $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\ln n}$ 14. $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^{3/2}}$ 15. $\sum_{n=1}^{\infty} \frac{1}{1 + \ln n}$

16. $\sum_{n=1}^{\infty} \frac{1}{(1 + \ln n)^2}$ 17. $\sum_{n=2}^{\infty} \frac{\ln(n + 1)}{n + 1}$ 18. $\sum_{n=1}^{\infty} \frac{1}{(1 + \ln^2 n)}$

19. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2 - 1}}$ 20. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2 + 1}$ 21. $\sum_{n=1}^{\infty} \frac{1 - n}{n2^n}$

2. diverges by the Direct Comparison Test since $n + n + n > n + \sqrt{n} + 0 \Rightarrow \frac{3}{n + \sqrt{n}} > \frac{1}{n}$, which is the nth term of the divergent series $\sum_{n=1}^{\infty} \frac{1}{n}$ or use Limit Comparison Test with $b_n = \frac{1}{n}$

14. converges by the Limit Comparison Test (part 2) with $\frac{1}{n^{5/4}}$, the nth term of a convergent p-series:

$$\lim_{n \to \infty} \frac{\left[\frac{(\ln n)^2}{n^{3/2}}\right]}{\left(\frac{1}{n^{5/4}}\right)} = \lim_{n \to \infty} \frac{(\ln n)^2}{n^{1/4}} = \lim_{n \to \infty} \frac{\left(\frac{2 \ln n}{n}\right)}{\left(\frac{1}{4n^{3/4}}\right)} = 8 \lim_{n \to \infty} \frac{\ln n}{n^{1/4}} = 8 \lim_{n \to \infty} \frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{4n^{3/4}}\right)} = 32 \lim_{n \to \infty} \frac{1}{n^{1/4}} = 32 \cdot 0 = 0$$

15. diverges by the Limit Comparison Test (part 3) with $\frac{1}{n}$, the nth term of the divergent harmonic series:

$$\lim_{n \to \infty} \frac{\left(\frac{1}{1 + \ln n}\right)}{\left(\frac{1}{n}\right)} = \lim_{n \to \infty} \frac{n}{1 + \ln n} = \lim_{n \to \infty} \frac{1}{\left(\frac{1}{n}\right)} = \lim_{n \to \infty} n = \infty$$

16. diverges by the Limit Comparison Test (part 3) with $\frac{1}{n}$, the nth term of the divergent harmonic series:

$$\lim_{n \to \infty} \frac{\left(\frac{1}{(1 + \ln n)^2}\right)}{\left(\frac{1}{n}\right)} = \lim_{n \to \infty} \frac{n}{(1 + \ln n)^2} = \lim_{n \to \infty} \frac{1}{\left[\frac{2(1 + \ln n)}{n}\right]} = \lim_{n \to \infty} \frac{n}{2(1 + \ln n)} = \lim_{n \to \infty} \frac{1}{\left(\frac{2}{n}\right)} = \lim_{n \to \infty} \frac{n}{2} = \infty$$

Which of the series in Exercises 1–26 converge, and which diverge? Give reasons for your answers. (When checking your answers, remember there may be more than one way to determine a series' convergence or divergence.)

1. $\sum_{n=1}^{\infty} \frac{n^{\sqrt{2}}}{2^n}$ 2. $\sum_{n=1}^{\infty} n^2 e^{-n}$

3. $\sum_{n=1}^{\infty} n! e^{-n}$ 4. $\sum_{n=1}^{\infty} \frac{n!}{10^n}$

5. $\sum_{n=1}^{\infty} \frac{n^{10}}{10^n}$ 6. $\sum_{n=1}^{\infty} \left(\frac{n - 2}{n}\right)^n$

7. $\sum_{n=1}^{\infty} \frac{2 + (-1)^n}{1.25^n}$ 8. $\sum_{n=1}^{\infty} \frac{(-2)^n}{3^n}$

9. $\sum_{n=1}^{\infty} \left(1 - \frac{3}{n}\right)^n$ 10. $\sum_{n=1}^{\infty} \left(1 - \frac{1}{3n}\right)^n$

11. $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ 12. $\sum_{n=1}^{\infty} \frac{(\ln n)^n}{n^n}$

13. $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)$ 14. $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)^n$

15. $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ 16. $\sum_{n=1}^{\infty} \frac{n \ln n}{2^n}$

17. $\sum_{n=1}^{\infty} \frac{(n + 1)(n + 2)}{n!}$ 18. $\sum_{n=1}^{\infty} e^{-n}(n^3)$

2. converges by the Ratio Test: $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{\left(\frac{(n+1)^2}{e^{n+1}}\right)}{\left(\frac{n^2}{e^n}\right)} = \lim_{n \to \infty} \frac{(n+1)^2}{e^{n+1}} \cdot \frac{e^n}{n^2} = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^2 \left(\frac{1}{e}\right) = \frac{1}{e} < 1$

4. diverges by the Ratio Test: $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{\left(\frac{(n+1)!}{10^{n+1}}\right)}{\left(\frac{n!}{10^n}\right)} = \lim_{n \to \infty} \frac{(n+1)!}{10^{n+1}} \cdot \frac{10^n}{n!} = \lim_{n \to \infty} \frac{n}{10} = \infty$

8. converges; a geometric series with $|r| = \left|-\frac{2}{3}\right| < 1$

16. converges by the Ratio Test: $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{(n+1)\ln(n+1)}{2^{n+1}} \cdot \frac{2^n}{n\ln(n)} = \frac{1}{2} < 1$

18. converges by the Ratio Test: $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{(n+1)^3}{e^{n+1}} \cdot \frac{e^n}{n^3} = \frac{1}{e} < 1$

In Exercises 45–48, estimate the magnitude of the error involved in using the sum of the first four terms to approximate the sum of the entire series.

45. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$

46. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{10^n}$

47. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(0.01)^n}{n}$

46. $|\text{error}| < \left|(-1)^6 \left(\frac{1}{10^5}\right)\right| = 0.00001$