42. $\displaystyle\int_0^1 \frac{dt}{t - \sin t}$ (*Hint:* $t \geq \sin t$ for $t \geq 0$)

43. $\displaystyle\int_0^2 \frac{dx}{1 - x^2}$ **44.** $\displaystyle\int_0^2 \frac{dx}{1 - x}$

42. $\int_0^1 \frac{dt}{t - \sin t}$; let $f(t) = \frac{1}{t - \sin t}$ and $g(t) = \frac{1}{t^3}$, then $\lim\limits_{t \to 0} \frac{f(t)}{g(t)} = \lim\limits_{t \to 0} \frac{t^3}{t - \sin t} = \lim\limits_{t \to 0} \frac{3t^2}{1 - \cos t} = \lim\limits_{t \to 0} \frac{6t}{\sin t}$ $= \lim\limits_{t \to 0} \frac{6}{\cos t} = 6$. Now, $\int_0^1 \frac{dt}{t^3} = \lim\limits_{b \to 0^+} \left[-\frac{1}{2t^2}\right]_b^1 = -\frac{1}{2} - \lim\limits_{b \to 0^+} \left[-\frac{1}{2b^2}\right] = +\infty$, which diverges $\Rightarrow \int_0^1 \frac{dt}{t - \sin t}$ diverges by the Limit Comparison Test.

23,25 and 31: Find Limit with proof

Each of Exercises 1–6 gives a formula for the nth term a_n of a sequence $\{a_n\}$. Find the values of a_1, a_2, a_3, and a_4.

1. $a_n = \frac{1 - n}{n^2}$ **2.** $a_n = \frac{1}{n!}$

3. $a_n = \frac{(-1)^{n+1}}{2n - 1}$ **4.** $a_n = 2 + (-1)^n$

5. $a_n = \frac{2^n}{2^{n+1}}$ **6.** $a_n = \frac{2^n - 1}{2^n}$

Each of Exercises 7–12 gives the first term or two of a sequence along with a recursion formula for the remaining terms. Write out the first ten terms of the sequence.

7. $a_1 = 1, \quad a_{n+1} = a_n + (1/2^n)$

8. $a_1 = 1, \quad a_{n+1} = a_n/(n + 1)$

9. $a_1 = 2, \quad a_{n+1} = (-1)^{n+1}a_n/2$

10. $a_1 = -2, \quad a_{n+1} = na_n/(n + 1)$

11. $a_1 = a_2 = 1, \quad a_{n+2} = a_{n+1} + a_n$

12. $a_1 = 2, \quad a_2 = -1, \quad a_{n+2} = a_{n+1}/a_n$

In Exercises 13–22, find a formula for the nth term of the sequence.

13. The sequence $1, -1, 1, -1, 1, \ldots$

14. The sequence $-1, 1, -1, 1, -1, \ldots$

15. The sequence $1, -4, 9, -16, 25, \ldots$

16. The sequence $1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, \ldots$

17. The sequence $0, 3, 8, 15, 24, \ldots$

18. The sequence $-3, -2, -1, 0, 1, \ldots$

19. The sequence $1, 5, 9, 13, 17, \ldots$

20. The sequence $2, 6, 10, 14, 18, \ldots$

21. The sequence $1, 0, 1, 0, 1, \ldots$

23. $a_n = 2 + (0.1)^n$

25. $a_n = \frac{1 - 2n}{1 + 2n}$

27. $a_n = \frac{1 - 5n^4}{n^4 + 8n^3}$

29. $a_n = \frac{n^2 - 2n + 1}{n - 1}$

31. $a_n = 1 + (-1)^n$

33. $a_n = \left(\frac{n + 1}{2n}\right)\left(1 - \frac{1}{n}\right)$

35. $a_n = \frac{(-1)^{n+1}}{2n - 1}$

37. $a_n = \sqrt{\frac{2n}{n + 1}}$

39. $a_n = \sin\left(\frac{\pi}{2} + \frac{1}{n}\right)$

41. $a_n = \frac{\sin n}{n}$

43. $a_n = \frac{n}{2^n}$

45. $a_n = \frac{\ln(n + 1)}{\sqrt{n}}$

47. $a_n = 8^{1/n}$

49. $a_n = \left(1 + \frac{7}{n}\right)^n$

51. $a_n = \sqrt[n]{10n}$

53. $a_n = \left(\frac{3}{n}\right)^{1/n}$

55. $a_n = \frac{\ln n}{n^{1/n}}$

14. $a_n = (-1)^n, n = 1, 2, \ldots$

16. $a_n = \frac{(-1)^{n+1}}{n^2}, n = 1, 2, \ldots$

18. $a_n = n - 4, n = 1, 2, \ldots$

20. $a_n = 4n - 2, n = 1, 2, \ldots$

23 The sequence has limit 2

$$a_n = 2 + \left(1/10\right)^n$$

Auxilliary calculation.

$$\left| a_n - 2 \right| = \left| 2 + \left(\frac{1}{10}\right)^n - 2 \right| = \left| \left(\frac{1}{10}\right)^n \right|$$

$= \left(\frac{1}{10}\right)^n$. We want this to be smaller than ε.

$\frac{1}{10}$ positive

Thus: Set $\varepsilon > \left(\frac{1}{10}\right)^n$ and solve for n:

$\Rightarrow 10^n > \frac{1}{\varepsilon} \underset{\lg \text{ is increasing}}{\Rightarrow} n > \lg_{10}\left(1/\varepsilon\right) = \dfrac{\lg(1) - \lg(\varepsilon)}{\lg(10)}$

$$= -\frac{\lg(\varepsilon)}{\lg(10)}$$

Thus: (This is the actual proof):

Given $\varepsilon > 0$, let $N = \dfrac{-\lg(\varepsilon)}{\lg(10)}$.

Then for $n > N$ we have that

$$\left| a_n - 2 \right| \underset{\text{as above}}{=} \ldots \left| \left(\frac{1}{10}\right)^n \right| = \left(\frac{1}{10}\right)^n \underset{\substack{n > N \\ \text{and } \frac{1}{10} < 1}}{<} \left(\frac{1}{10}\right)^N$$

$$= \left(\frac{1}{10}\right)^{-\frac{\lg(\varepsilon)}{\lg(10)}} = 10^{\left(\lg(\varepsilon)/\lg(10)\right)} = \varepsilon.$$

Thus $\lim_{n \to \infty} a_n = 2$

25] Claim: The sequence has limit -1 (evaluate for big n as do the $\lim\limits_{x\to\infty} \frac{1-2x}{1+2x}$)

Auxillary Calculation: $\left|a_n-(-1)\right| = \left|\frac{1-2n}{1+2n}+1\right|$

$$= \left|\frac{1-2n+1+2n}{1+2n}\right| = \left|\frac{2}{1+2n}\right| \underset{1+2n>2n}{<} \left|\frac{2}{2n}\right| = \left|\frac{1}{n}\right| = \frac{1}{n}.$$

We want this to be $< \varepsilon$, thus setting $\varepsilon > \frac{1}{n}$, solve for n, we get: $n > \frac{1}{\varepsilon}$

Thus (actual proof of limit):

Given $\varepsilon > 0$, let $N = 1/\varepsilon$. Then for $n > N$:

$$\left|a_n-(-1)\right| \underset{\text{as above}}{=} \cdots \left|\frac{2}{1+2n}\right| < \left|\frac{2}{2n}\right| = \frac{1}{n} \underset{n>N}{<} \frac{1}{N} = \frac{1}{1/\varepsilon} = \varepsilon$$

Thus $\lim\limits_{n\to\infty} a_n = -1$.

31] This sequence has no limit, the argument is similar as Example 2 in the book.