$\ln x\, dx$ 6. $\int_1 x^3 \ln x\, dx$

$n^{-1} y\, dy$ 8. $\int \sin^{-1} y\, dy$

$\sec^2 x\, dx$ 10. $\int 4x \sec^2 2x\, dx$

8. $u = \sin^{-1} y$, $du = \frac{dy}{\sqrt{1-y^2}}$; $dv = dy$, $v = y$;

$\int \sin^{-1} y\, dy = y \sin^{-1} y - \int \frac{y\, dy}{\sqrt{1-y^2}} = y \sin^{-1} y + \sqrt{1-y^2} + C$

96. Using different substitutions Show that the integral

$$\int ((x^2 - 1)(x + 1))^{-2/3}\, dx$$

can be evaluated with any of the following substitutions.

a. $u = 1/(x + 1)$

b. $u = ((x - 1)/(x + 1))^k$ for $k = 1, 1/2, 1/3, -1/3, -2/3$, and -1

c. $u = \tan^{-1} x$

d. $u = \tan^{-1} \sqrt{x}$

e. $u = \tan^{-1}((x - 1)/2)$

f. $u = \cos^{-1} x$

g. $u = \cosh^{-1} x$

What is the value of the integral? (*Source:* "Problems and Solutions," *College Mathematics Journal*, Vol. 21, No. 5 (Nov. 1990), pp. 425–426.)

96. $\left[(x^2-1)(x+1)\right]^{-2/3} = \left[(x-1)(x+1)^2\right]^{-2/3} = (x-1)^{-2/3}(x+1)^{-4/3} = (x+1)^{-2}\left[(x-1)^{-2/3}(x+1)^{2/3}\right]$

$= (x+1)^{-2}\left(\frac{x-1}{x+1}\right)^{-2/3} = (x+1)^{-2}\left(1-\frac{2}{x+1}\right)^{-2/3}$

(a) $\int \left[(x^2-1)(x+1)\right]^{-2/3} dx = \int (x+1)^{-2}\left(1-\frac{2}{x+1}\right)^{-2/3} dx;\ \begin{bmatrix} u = \frac{1}{x+1} \\ du = -\frac{1}{(x+1)^2}\,dx \end{bmatrix}$

$\rightarrow \int -(1-2u)^{-2/3}\,du = \frac{3}{2}(1-2u)^{1/3} + C = \frac{3}{2}\left(1-\frac{2}{x+1}\right)^{1/3} + C = \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{1/3} + C$

(b) $\int \left[(x^2-1)(x+1)\right]^{-2/3} dx = \int (x+1)^{-2}\left(\frac{x-1}{x+1}\right)^{-2/3} dx;\ u = \left(\frac{x-1}{x+1}\right)^{k}$

$\Rightarrow du = k\left(\frac{x-1}{x+1}\right)^{k-1}\frac{[(x+1)-(x-1)]}{(x+1)^2}\,dx = 2k\,\frac{(x-1)^{k-1}}{(x+1)^{k+1}}\,dx;\ dx = \frac{(x+1)^2}{2k}\left(\frac{x+1}{x-1}\right)^{k-1} du$

$= \frac{(x+1)^2}{2k}\left(\frac{x-1}{x+1}\right)^{1-k} du;$ then, $\int \left(\frac{x-1}{x+1}\right)^{-2/3}\frac{1}{2k}\left(\frac{x-1}{x+1}\right)^{1-k} du = \frac{1}{2k}\int \left(\frac{x-1}{x+1}\right)^{(1/3-k)} du$

$= \frac{1}{2k}\int \left(\frac{x-1}{x+1}\right)^{k(1/3k-1)} du = \frac{1}{2k}\int u^{(1/3k-1)}\,du = \frac{1}{2k}(3k)\,u^{1/3k} + C = \frac{3}{2}u^{1/3k} + C = \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{1/3} + C$

(c) $\int \left[(x^2-1)(x+1)\right]^{-2/3} dx = \int (x+1)^{-2}\left(\frac{x-1}{x+1}\right)^{-2/3} dx;$

$\begin{bmatrix} u = \tan^{-1} x \\ x = \tan u \\ dx = \frac{du}{\cos^2 u} \end{bmatrix} \rightarrow \int \frac{1}{(\tan u+1)^2}\left(\frac{\tan u - 1}{\tan u + 1}\right)^{-2/3}\left(\frac{du}{\cos^2 u}\right) = \int \frac{1}{(\sin u + \cos u)^2}\left(\frac{\sin u - \cos u}{\sin u + \cos u}\right)^{-2/3} du;$

$\begin{bmatrix} \sin u + \cos u = \sin u + \sin\left(\frac{\pi}{2}-u\right) = 2\sin\frac{\pi}{4}\cos\left(u-\frac{\pi}{4}\right) \\ \sin u - \cos u = \sin u - \sin\left(\frac{\pi}{2}-u\right) = 2\cos\frac{\pi}{4}\sin\left(u-\frac{\pi}{4}\right) \end{bmatrix} \rightarrow \int \frac{1}{2\cos^2\left(u-\frac{\pi}{4}\right)}\left[\frac{\sin\left(u-\frac{\pi}{4}\right)}{\cos\left(u-\frac{\pi}{4}\right)}\right]^{-2/3} du$

$= \frac{1}{2}\int \tan^{-2/3}\left(u-\frac{\pi}{4}\right)\sec^2\left(u-\frac{\pi}{4}\right) du = \frac{3}{2}\tan^{1/3}\left(u-\frac{\pi}{4}\right) + C = \frac{3}{2}\left[\frac{\tan u - \tan\frac{\pi}{4}}{1+\tan u\tan\frac{\pi}{4}}\right]^{1/3} + C$

$= \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{1/3} + C$

(d) $u = \tan^{-1}\sqrt{x} \Rightarrow \tan u = \sqrt{x} \Rightarrow \tan^2 u = x \Rightarrow dx = 2\tan u\left(\frac{1}{\cos^2 u}\right) du = \frac{2\sin u}{\cos^3 u}\,du = -\frac{2d(\cos u)}{\cos^3 u};$

$x - 1 = \tan^2 u - 1 = \frac{\sin^2 u - \cos^2 u}{\cos^2 u} = \frac{1-2\cos^2 u}{\cos^2 u};\ x + 1 = \tan^2 u + 1 = \frac{\cos^2 u + \sin^2 u}{\cos^2 u} = \frac{1}{\cos^2 u};$

$\int (x-1)^{-2/3}(x+1)^{-4/3}\,dx = \int \frac{(1-2\cos^2 u)^{-2/3}}{(\cos^2 u)^{-2/3}} \cdot \frac{1}{(\cos^2 u)^{-4/3}} \cdot \frac{-2d(\cos u)}{\cos^3 u}$

$= \int (1-2\cos^2 u)^{-2/3} \cdot (-2) \cdot \cos u \cdot d(\cos u) = \frac{1}{2}\int (1-2\cos^2 u)^{-2/3} \cdot d(1-2\cos^2 u)$

$= \frac{3}{2}(1-2\cos^2 u)^{1/3} + C = \frac{3}{2}\left[\frac{\left(\frac{1-2\cos^2 u}{\cos^2 u}\right)}{\left(\frac{1}{\cos^2 u}\right)}\right]^{1/3} + C = \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{1/3} + C$

(e) $u = \tan^{-1}\left(\frac{x-1}{2}\right) \Rightarrow \frac{x-1}{2} = \tan u \Rightarrow x + 1 = 2(\tan u + 1) \Rightarrow dx = \frac{2\,du}{\cos^2 u} = 2d(\tan u);$

$\int (x-1)^{-2/3}(x+1)^{-4/3}\,dx = \int (\tan u)^{-2/3}(\tan u + 1)^{-4/3} \cdot 2^{-2} \cdot 2 \cdot d(\tan u)$

$= \frac{1}{2}\int \left(1-\frac{1}{\tan u + 1}\right)^{-2/3} d\left(1-\frac{1}{\tan u + 1}\right) = \frac{3}{2}\left(1-\frac{1}{\tan u + 1}\right)^{1/3} + C = \frac{3}{2}\left(1-\frac{2}{x+1}\right)^{1/3} + C$

$= \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{1/3} + C$

(f) $\begin{bmatrix} u = \cos^{-1} x \\ x = \cos u \\ dx = -\sin u\,du \end{bmatrix} \rightarrow -\int \frac{\sin u\,du}{\sqrt[3]{(\cos^2 u - 1)^2(\cos u + 1)^2}} = -\int \frac{\sin u\,du}{(\sin^{4/3} u)\left(2^{2/3}\cos\frac{u}{2}\right)^{4/3}}$

$= -\int \frac{du}{(\sin u)^{1/3}\left(2^{2/3}\cos\frac{u}{2}\right)^{4/3}} = -\int \frac{du}{2\left(\sin\frac{u}{2}\right)^{1/3}\left(\cos\frac{u}{2}\right)^{5/3}} = -\frac{1}{2}\int \left(\frac{\cos\frac{u}{2}}{\sin\frac{u}{2}}\right)^{1/3}\frac{du}{\left(\cos^2\frac{u}{2}\right)}$

$= -\int \tan^{-1/3}\left(\frac{u}{2}\right) d\left(\tan\frac{u}{2}\right) = -\frac{3}{2}\tan^{2/3}\frac{u}{2} + C = \frac{3}{2}\left(-\tan^2\frac{u}{2}\right)^{1/3} + C = \frac{3}{2}\left(\frac{\cos u - 1}{\cos u + 1}\right)^{1/3} + C$

$= \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{1/3} + C$

(g) $\int \left[(x^2-1)(x+1)\right]^{-2/3} dx;\ \begin{bmatrix} u = \cosh^{-1} x \\ x = \cosh u \\ dx = \sinh u \end{bmatrix} \rightarrow \int \frac{\sinh u\,du}{\sqrt[3]{(\cosh^2 u - 1)^2(\cosh u + 1)^2}}$

$= \int \frac{\sinh u\,du}{\sqrt[3]{(\sinh^4 u)(\cosh u + 1)^2}} = \int \frac{du}{\sqrt[3]{(\sinh u)\left(4\cosh^4\frac{u}{2}\right)}} = \frac{1}{2}\int \frac{du}{\sqrt[3]{\sinh\left(\frac{u}{2}\right)\cosh^5\left(\frac{u}{2}\right)}}$

$= \int \left(\tanh\frac{u}{2}\right)^{-1/3} d\left(\tanh\frac{u}{2}\right) = \frac{3}{2}\left(\tanh\frac{u}{2}\right)^{2/3} + C = \frac{3}{2}\left(\frac{\cosh u - 1}{\cosh u + 1}\right)^{1/3} + C = \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{1/3} + C$