40. \( y = x^{x+1} \Rightarrow \ln y = \ln x^{x+1} = (x + 1) \ln x \Rightarrow \frac{y'}{y} = \ln x + (x + 1) \left(\frac{1}{x}\right) = \ln x + 1 + \frac{1}{x} \Rightarrow y' = x^{x+1} \left(1 + \frac{1}{x} + \ln x\right)\)

Solve the equations in Exercises 7–10 for \( x \).

7. \( \ln(x+1) = \ln x^2 \Rightarrow x+1 = x^2 \Rightarrow x^2 - x - 1 = 0 \)

8. \( \log_2(x) = \log_3(x^2) \Rightarrow \log_2 x = \log_3 x^2 = 2 \log_3 x \)

9. \( \log_3(x^3) - 3 \log_2 x = 3 \log_3 (2) \)

10. \( \ln e + 4 \cdot 2 \log_3 (100) = \frac{1}{3} \log_{10} (100) \)

Logarithmic Differentiation

In Exercises 39–42, use logarithmic differentiation to find the derivative of \( y \) with respect to the given independent variable.

39. \( y = (x + 1)^3 \)

40. \( y = x^{x+1} \)

41. \( y = (\sqrt{x})^y \)

42. \( y = x^x \)

(a) The point of tangency is \( (p, \ln p) \) and \( m_{\text{tangent}} = \frac{1}{p} \) since \( \frac{dy}{dx} = \frac{1}{x} \). The tangent line passes through \( (0, 0) \Rightarrow \) the equation of the tangent line is \( y = \frac{1}{p} x \). The tangent line also passes through \( (p, \ln p) \Rightarrow \ln p = \frac{1}{p} p = 1 \Rightarrow p = e \), and the tangent line equation is \( y = \frac{1}{e} x \).

(b) \( \frac{dy}{dx} = -\frac{1}{e} \) for \( x \neq 0 \Rightarrow y = \ln x \) is concave downward over its domain. Therefore, \( y = \ln x \) lies below the graph of \( y = \frac{1}{e} x \) for all \( x > 0 \), \( x \neq e \), and \( \ln x < \frac{1}{e} x \) for \( x > 0 \), \( x \neq e \).

(c) Multiplying by \( e \), \( \ln x < x \) or \( \ln e^x < x \).

(d) Exponentiating both sides of \( \ln e^x < x \), we have \( e^{\ln e^x} < e^x \), or \( e^x < e^x \) for all positive \( x \neq e \).

(e) Let \( x = \pi \) to see that \( \pi^e < e^\pi \). Therefore, \( e^\pi \) is bigger.
Webwork 9.1/Problem 3

Solve the differential equation

\[ \frac{du}{dt} = e^{2u+5t}. \]

for the initial condition \( u(0) = 7 \). (In your case the numbers may have differed.)

We separate the variables: \( e^{-2u}du = e^{5t}dt \) and integrate (already bringing both constants to one side):

\[ -\frac{1}{2}e^{-2u} = \frac{1}{5}e^{5t} + C. \]

Solving for \( u \) yields

\[ u = -\frac{1}{2} \ln \left( -2 \cdot \left( \frac{1}{5}e^{5t} + C \right) \right) \]

Do not worry about the \(-2\) in the argument for the logarithm – we will have \( C \) a big negative number such that the argument of the logarithm is positive!

Now we use the initial condition to get the value for \( C \). Setting \( t = 0 \) yields:

\[ 7 = u(0) = -\frac{1}{2} \ln \left( -2 \cdot \left( \frac{1}{5}e^{5 \cdot 0} + C \right) \right) \]

and therefore

\[ C = -\frac{1}{2}e^{-2 \cdot 7} - \frac{1}{5}e^{5 \cdot 0} = -\frac{1}{2}e^{-14} - \frac{1}{5}. \]

Thus the solution we want is:

\[ u = -\frac{1}{2} \ln \left( -2 \cdot \left( \frac{1}{5}e^{5t} - \frac{1}{2}e^{-14} - \frac{1}{5} \right) \right). \]

We see from this solution that for \( t > \frac{1}{5} \cdot \ln \left( \frac{5}{2e^{14}} + 1 \right) \sim 4.1 \cdot 10^{-7} \) the argument of the logarithm becomes negative. This means that for larger \( t \) values the differential equation has no solution.

This is not surprising if we plot the solution — we see that at this \( t \)-value the solution curve goes to \( \infty \).

If the differential equation came from a model in physics, this essentially means that the model does not work for larger \( t \)-values for the given initial configuration \( u(0) = 7 \). (For example at this point the machine will have been broken or the particle has moved away or hit a wall.)