Find the first four terms of the binomial series for the functions in Exercises 1–10.

1. $(1+x)^{1/2}$
2. $(1+x)^{1/3}$
3. $(1-x)^{-1/2}$
4. $(1-2x)^{1/2}$
5. $\left(1+\frac{x}{2}\right)^{-2}$
6. $\left(1-\frac{x}{2}\right)^{-2}$
9. $\left(1+\frac{1}{x}\right)^{1/2}$
10. $\left(1-\frac{2}{x}\right)^{1/3}$

Find the binomial series for the functions in Exercises 11–14.

11. $(1+x)^4$
12. $(1+x^2)^3$

Find series solutions for the initial value problems in Exercises 15–32.

15. $y' + y = 0, \quad y(0) = 1$
16. $y' - 2y = 0, \quad y(0) = 1$
17. $y' - y = 1, \quad y(0) = 0$
18. $y' + y = 1, \quad y(0) = 2$
19. $y' - y = x, \quad y(0) = 0$
20. $y' + y = 2x, \quad y(0) = -1$
21. $y' - xy = 0, \quad y(0) = 1$
22. $y' - x^2y = 0, \quad y(0) = 1$
23. $(1-x)y' - y = 0, \quad y(0) = 2$
24. $(1+x^2)y' + 2xy = 0, \quad y(0) = 3$
25. $y'' - y = 0, \quad y'(0) = 1$ and $y(0) = 0$
26. $y'' + y = 0, \quad y'(0) = 0$ and $y(0) = 1$
27. $y'' + y = x, \quad y'(0) = 1$ and $y(0) = 2$
28. $y'' - y = x, \quad y'(0) = 2$ and $y(0) = -1$
29. $y'' - y = -x, \quad y'(2) = -2$ and $y(2) = 0$
30. $y'' - x^2y = 0, \quad y'(0) = b$ and $y(0) = a$
31. $y'' + x^2y = x, \quad y'(0) = b$ and $y(0) = a$
32. $y'' - 2y' + y = 0, \quad y'(0) = 1$ and $y(0) = 0$

Approximations and Nonelementary Integrals

In Exercises 33–36, use series to estimate the integrals' values with an error of magnitude less than 10^{-3}. (The answer section gives the integrals' values rounded to five decimal places.)

33. $\int_0^{0.2} \sin x^2\, dx$
34. $\int_0^{0.2} \frac{e^{-x}-1}{x}\, dx$
35. $\int_0^{0.1} \frac{1}{\sqrt{1+x^4}}\, dx$
36. $\int_0^{0.25} \sqrt[3]{1+x^2}\, dx$

4. $(1-2x)^{1/2} = 1 + \frac{1}{2}(-2x) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)(-2x)^2}{2!} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-2x)^3}{3!} + \ldots = 1 - x - \frac{1}{2}x^2 - \frac{1}{2}x^3 - \ldots$

6. $\left(1-\frac{x}{2}\right)^{-2} = 1 - 2\left(-\frac{x}{2}\right) + \frac{(-2)(-3)\left(-\frac{x}{2}\right)^2}{2!} + \frac{(-2)(-3)(-4)\left(-\frac{x}{2}\right)^3}{3!} + \ldots = 1 + x + \frac{3}{4}x^2 + \frac{1}{2}x^3 + \ldots$

12. $(1+x^2)^3 = 1 + 3x^2 + \frac{(3)(2)(x^2)^2}{2!} + \frac{(3)(2)(1)(x^2)^3}{3!} = 1 + 3x^2 + 3x^4 + x^6$

$n=0$

18. Assume the solution has the form $y = a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1} + a_nx^n + \dots$

$\Rightarrow \frac{dy}{dx} = a_1 + 2a_2x + \dots + na_nx^{n-1} + \dots$

$\Rightarrow \frac{dy}{dx} + y = (a_1 + a_0) + (2a_2 + a_1)x + (3a_3 + a_2)x^2 + \dots + (na_n + a_{n-1})x^{n-1} + \dots = 1$

$\Rightarrow a_1 + a_0 = 1,\ 2a_2 + a_1 = 0,\ 3a_3 + a_2 = 0$ and in general $na_n + a_{n-1} = 0$. Since $y = 2$ when $x = 0$ we have $a_0 = 2$. Therefore $a_1 = 1 - a_0 = -1,\ a_2 = \frac{-a_1}{2\cdot 1} = \frac{1}{2},\ a_3 = \frac{-a_2}{3} = -\frac{1}{3\cdot 2},\ \dots,\ a_n = \frac{-a_{n-1}}{n} = \frac{(-1)^n}{n!}$

$\Rightarrow y = 2 - x + \frac{1}{2}x^2 - \frac{1}{3\cdot 2}x^3 + \dots + \frac{(-1)^n}{n!}x^n + \dots = 1 + \left(1 - x + \frac{1}{2}x^2 - \frac{1}{3\cdot 2}x^3 + \dots + \frac{(-1)^n}{n!}x^n + \dots\right)$

$= 1 + \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} = 1 + e^{-x}$

22. $y' - x^2y = a_1 + 2a_2x + (3a_3 - a_0)x^2 + (4a_4 - a_1)x^3 + \dots + (na_n - a_{n-3})x^{n-1} + \dots = 0 \Rightarrow a_1 = 0,\ a_2 = 0,$ $3a_3 - a_0 = 0,\ 4a_4 - a_1 = 0$ and in general $na_n - a_{n-3} = 0$. Since $y = 1$ when $x = 0$, we have $a_0 = 1$. Therefore $a_3 = \frac{a_0}{3} = \frac{1}{3},\ a_4 = \frac{a_1}{4} = 0,\ a_5 = \frac{a_2}{5} = 0,\ a_6 = \frac{a_3}{6} = \frac{1}{3\cdot 6},\ \dots,\ a_{3n} = \frac{1}{3\cdot 6\cdot 9\cdots 3n},\ a_{3n+1} = 0$ and $a_{3n+2} = 0$

$\Rightarrow y = 1 + \frac{1}{3}x^3 + \frac{1}{3\cdot 6}x^6 + \frac{1}{3\cdot 6\cdot 9}x^9 + \dots + \frac{1}{3\cdot 6\cdot 9\cdots 3n}x^{3n} + \dots = \sum_{n=0}^{\infty} \frac{x^{3n}}{3^n n!} = \sum_{n=0}^{\infty} \frac{\left(\frac{x^3}{3}\right)^n}{n!} = e^{x^3/3}$

34. $\int_0^{0.2} \frac{e^{-x} - 1}{x}\,dx = \int_0^{0.2} \frac{1}{x}\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots - 1\right)dx = \int_0^{0.2}\left(-1 + \frac{x}{2} - \frac{x^2}{6} + \frac{x^3}{24} - \dots\right)dx$

$= \left[-x + \frac{x^2}{4} - \frac{x^3}{18} + \dots\right]_0^{0.2} \approx -0.19044$ with error $|E| \le \frac{(0.2)^4}{96} \approx 0.00002$

finding a Cartesian equation for it. Graph the Cartesian equation. (The graphs will vary with the equation used.) Indicate the portion of the graph traced by the particle and the direction of motion.

67. $x = \cos 2t, \quad y = \sin 2t, \quad 0 \le t \le \pi$

68. $x = \cos(\pi - t), \quad y = \sin(\pi - t), \quad 0 \le t \le \pi$

69. $x = 4\cos t, \quad y = 2\sin t, \quad 0 \le t \le 2\pi$

70. $x = 4\sin t, \quad y = 5\cos t, \quad 0 \le t \le 2\pi$

71. $x = 3t, \quad y = 9t^2, \quad -\infty < t < \infty$

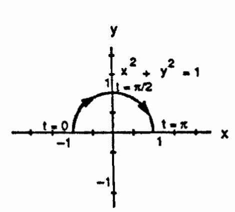
72. $x = -\sqrt{t}, \quad y = t, \quad t \ge 0$

73. $x = 2t - 5, \quad y = 4t - 7, \quad -\infty < t < \infty$

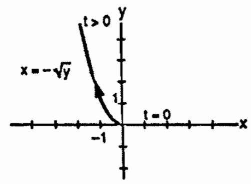
68. $x = \cos(\pi - t),\ y = \sin(\pi - t),\ 0 \le t \le \pi$

$\Rightarrow \cos^2(\pi - t) + \sin^2(\pi - t) = 1$

$\Rightarrow x^2 + y^2 = 1,\ y \ge 0$



72. $x = -\sqrt{t},\ y = t,\ t \geq 0 \Rightarrow x = -\sqrt{y}$
or $y = x^2,\ x \leq 0$



Find the lengths of the curves in Exercises 1–6.

1. $x = 1 - t, \quad y = 2 + 3t, \quad -2/3 \leq t \leq 1$

2. $x = \cos t, \quad y = t + \sin t, \quad 0 \leq t \leq \pi$

3. $x = t^3, \quad y = 3t^2/2, \quad 0 \leq t \leq \sqrt{3}$

4. $x = t^2/2, \quad y = (2t + 1)^{3/2}/3, \quad 0 \leq t \leq 4$

2. $\frac{dx}{dt} = -\sin t$ and $\frac{dy}{dt} = 1 + \cos t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(-\sin t)^2 + (1 + \cos t)^2} = \sqrt{2 + 2\cos t}$
$\Rightarrow \text{Length} = \int_0^\pi \sqrt{2 + 2\cos t}\, dt = \sqrt{2}\int_0^\pi \sqrt{\left(\frac{1 - \cos t}{1 - \cos t}\right)(1 + \cos t)}\, dt = \sqrt{2}\int_0^\pi \sqrt{\frac{\sin^2 t}{1 - \cos t}}\, dt$
$= \sqrt{2}\int_0^\pi \frac{\sin t}{\sqrt{1 - \cos t}}\, dt$ (since $\sin t \geq 0$ on $[0, \pi]$); $[u = 1 - \cos t \Rightarrow du = \sin t\, dt;\ t = 0 \Rightarrow u = 0,$
$t = \pi \Rightarrow u = 2] \rightarrow \sqrt{2}\int_0^2 u^{-1/2}\, du = \sqrt{2}\left[2u^{1/2}\right]_0^2 = 4$