Additional Notes on Power Series

Manuela Girotti

MATH 317-01 Advanced Calculus of one variable

Contents

1 Quick recall 1
2 Abel’s Theorem 2
3 Differentiation and Integration of Power series 4

1 Quick recall

We recall here the main facts about power series. A detailed description of the topic has been covered a few weeks ago and it refers to the Section 2.6.5 of the textbook.

Definition 1. Let \(\{a_n\}\) be a sequence of real numbers and \(c \in \mathbb{R}\). The power series centered at \(c\) with coefficients \(\{a_n\}\) is the function

\[
\sum_{n=0}^{+\infty} a_n (x - c)^n.
\]

Every power series is a function defined on a certain domain (called interval of convergence) that is identified by the radius of convergence \(R\):

Theorem 2. Let

\[
f(x) := \sum_{n=0}^{+\infty} a_n (x - c)^n
\]

be a power series. There exists a constant \(R \in \mathbb{R} \cup \{+\infty\}\) such that the series converges absolutely for \(|x - c| < R\) (i.e. for any \(x \in (c - R, c + R)\)) and diverges for \(|x - c| > R\).

Theorem 3 (Hadamard’s Theorem). The radius of convergence \(R\) of the power series \(f(x) := \sum_{n=0}^{+\infty} a_n (x - c)^n\) is given by

\[
R = \limsup_{n \to +\infty} \sqrt[n]{|a_n|}.
\]

Remark 4. The radius of convergence of a power series can also be calculated as

\[
R = \frac{1}{\lim_{n \to +\infty} \left| \frac{a_{n+1}}{a_n} \right|}.
\]
2 Abel’s Theorem

From the theorems above we have that given a power series, the series converges absolutely for any point \( x \) in the open interval \((c - R, c + R)\), however no information is given about what happens at the endpoints \( x = c \pm R \) and in general the series may converges or diverges there.

On the other end, suppose that we have a power series (say, centered at \( c = 0 \))

\[
f(x) := \sum_{n=0}^{+\infty} a_n x^n
\]

with radius of convergence \( R \) (therefore the series converges for any \( x \in (-R, R) \)) and suppose that for example the series \( \sum_{n=0}^{\infty} a_n R^n < +\infty \). How do we know that

\[
f(R) = \sum_{n=0}^{\infty} a_n R^n
\]

(i.e. we just plug the value \( R \) into the definition of the power series)

To state the problem in a more correct way, we need to prove that assuming that \( \sum_{n=0}^{\infty} a_n R^n < +\infty \), then

\[
\lim_{x \to R} f(x) = \sum_{n=0}^{\infty} a_n R^n.
\]

The following theorem has a positive answers to this problem. We are considering here the simple case where the center of the series is \( c = 0 \), the radius of convergence is \( R = 1 \) and we’ll be checking only the right end point of the interval of convergence, namely \( x = 1 \).

The result can be easily generalized to any power series centered at a point \( c \in \mathbb{R} \) with radius of convergence \( R > 0 \) and with \( x \) equal either one of the endpoints of the interval of convergence.

**Theorem 5 (Abel’s Theorem).** Let \( f(x) := \sum_{n=0}^{+\infty} a_n x^n \) a power series with radius of convergence \( R = 1 \). If \( \sum_{n=0}^{\infty} a_n < +\infty \), then

\[
\lim_{x \to 1} f(x) = \sum_{n=0}^{\infty} a_n.
\]

**Proof.** The main tool used here will be the **summation by parts formula** for two finite sequences \( \{u_1, \ldots, u_n\} \) and \( \{v_0, \ldots, v_n\} \):

\[
\sum_{k=1}^{n} u_k (v_k - v_{k-1}) = u_n v_n - u_1 v_0 - \sum_{k=1}^{n-1} v_k (u_{k+1} - u_k)
\]

1. We know that \( \sum_{n=0}^{\infty} a_n < +\infty \), meaning that the sequence of partial sums \( \{s_n := \sum_{k=0}^{n} a_n\} \) converges \( (s_n \to S) \). We recall also that \( a_n = s_n - s_{n-1} \) for all \( n \in \mathbb{N} \) and \( a_0 = s_0 \).

Now, we will use the summation by parts formula for \( u_k = x^k \) and \( v_k = s_k \)

\[
\sum_{k=0}^{n} a_k x^k = a_0 + \sum_{k=1}^{n} (s_k - s_{k-1}) x^k = a_0 + s_n x^n - s_0 x - \sum_{k=1}^{n-1} s_k (x^{k+1} - x^k)
\]

\[
= a_0 (1 - x) + s_n x^n + \sum_{k=1}^{n-1} s_k (1 - x) x^k = s_n x^n + \sum_{k=0}^{n-1} s_k (1 - x) x^k
\]

2
This chain of equality is always valid and additionally we know that for any $|x| < 1$ the left hand side of the identity is (absolutely) convergent. Also, the term $s_n x^n \to 0$ as $n \to +\infty$, because $x^n \to 0$ ($|x| < 1$) and the sequence of partial sums $\{s_n\}$ of the (numeric series) $\sum_n a_n$ is convergent, therefore it is bounded.

Taking the limit as $k \to +\infty$ on both sides, we have

$$\sum_{k=0}^{\infty} a_k x^k = (1 - x) \sum_{k=0}^{\infty} s_k x^k.$$  

2. If $S = \sum_{n=0}^{\infty} a_n = \lim_{n \to +\infty} s_n$ is the sum of the (numeric) series, we want to show that $\sum_{n=0}^{\infty} a_n x^n \to S$ as $x \nearrow 1$: first of all

$$\sum_{n=0}^{\infty} a_n x^n - S = (1 - x) \sum_{n=0}^{\infty} s_n x^n - S = (1 - x) \sum_{n=0}^{\infty} (s_n - S) x^n$$

(the second equality comes from the fact that $(1 - x) \sum_{n=0}^{\infty} x^n = (1 - x) \frac{1}{1-x} = 1$, geometric series).

3. Then, since $s_n \to S$, this means that $\forall \; \epsilon > 0 \; \exists \; M_\epsilon \in \mathbb{N}$ such that $|s_n - S| < \epsilon$ for $n \geq M_\epsilon$; so, we fix $\epsilon > 0$ and we split the series (the first term will be a just a finite sum, therefore bounded)

$$\left| (1 - x) \sum_{n=0}^{\infty} (s_n - S) x^n \right| \leq (1 - x) \sum_{n=0}^{M_\epsilon-1} |s_n - S| |x|^n + (1 - x) \sum_{n=M_\epsilon}^{\infty} |s_n - S| |x|^n \leq (1 - x) K + (1 - x) \epsilon \sum_{n=0}^{\infty} |x|^n = (1 - x) K + (1 - x) \epsilon \frac{1}{1 - |x|}$$

where we don’t write $|1 - x|$ because $|x| < 1$; also, since we want to take the limit as $x \nearrow 1$, we can assume that $0 < x < 1$. In conclusion,

$$\left| (1 - x) \sum_{n=0}^{\infty} (s_n - S) x^n \right| \leq K(1 - x) + \epsilon \to 0.$$

Example 1. The series

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x - 1)^n$$

has radius of convergence

$$R = \frac{1}{\limsup_{n \to +\infty} \sqrt[n]{\frac{|(-1)^{n+1}|}{n}}} = \frac{1}{\limsup_{n \to +\infty} n^{-\frac{1}{n}}} = 1$$
so it converges for $|x - 1| < 1$ and it diverges for $|x - 1| > 1$. We still need to check the endpoints, i.e. $x = 0$ and $x = 2$.

For $x = 2$ the power series becomes an alternating harmonic series $f(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ which converges thanks to Leibniz’s rule. For $x = 0$, the series become

$$f(0) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(-1)^n = -\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n} = -\sum_{n=1}^{\infty} \frac{1}{n}$$

which is a harmonic series and therefore diverges.

In conclusion, the interval of convergence is $(0, 2]$.

**Example 2.** The series

$$f(x) = \sum_{n=0}^{\infty} (-1)^n x^{2^n} = x - x^2 + x^4 - x^8 + x^{16} - x^{32} + \ldots$$

has coefficients

$$a_n = \begin{cases} 1 & \text{if } n = 2^k \\ 0 & \text{if } n \neq 2^k \end{cases}$$

and its radius of convergence is equal to

$$R = \limsup_{n \to +\infty} \sqrt[n]{|a_n|} = 1$$

so it converges for $|x| < 1$ and it diverges for $|x| > 1$.

At the endpoints we have that $f(1) = f(-1) = \sum_{n=0}^{\infty} (-1)^n$ which is clearly a divergent series. Therefore the interval of convergence is $(-1, 1)$.

**Note 6.** This series is called **lacunary power series** because there are successively longer gaps (“lacuna”) between the powers with non-zero coefficients.

### 3 Differentiation and Integration of Power series

We will now tackle the problem of differentiating and integrating power series.

**Theorem 7 (Term-by-term differentiation of a power series).** Suppose that the power series

$$f(x) = \sum_{n=0}^{\infty} a_n(x - c)^n$$

has radius of convergence $R$. Then the power series

$$g(x) = \sum_{n=1}^{\infty} na_n(x - c)^{n-1}$$

is well defined and it also has radius of convergence $R$.

Moreover, the power series $f$ is differentiable $\forall x \in (c - R, c + R)$ and we have the identity

$$f'(x) = g(x) = \sum_{n=1}^{\infty} na_n(x - c)^{n-1}.$$
We will see only the proof of the first part of the theorem here. To prove that the power series $g$ is indeed the derivative of the power series $f$ some extra tools from Analysis are required, that we don’t possess at the moment.

**Proof.** Assume without loss of generality that $c = 0$. Let $|x| < R$: this implies (least upper bound properties of the set $\mathbb{R}$) that there exists $\rho > 0$ such that $|x| < \rho < R$ and let

$$r := \frac{|x|}{\rho} \in (0, 1).$$

To estimate the terms in the differentiated power series by the terms in the original series, we write

$$|na_n x^{n-1}| = \left| \frac{n}{\rho} \left( \frac{|x|}{\rho} \right)^{n-1} a_n \right| \rho^n = \frac{nr^{n-1}}{\rho} |a_n| \rho^n$$

The series $\sum_{n=1}^{\infty} nr^{n-1}$ converges by ratio test, since

$$\lim_{n \to \infty} \frac{(n+1)r^n}{nr^{n-1}} = \lim_{n \to \infty} r \left(1 + \frac{1}{n}\right) = r < 1$$

therefore the sequence $\{nr^{n-1}\}$ is bounded (it actually goes to zero): $-M \leq nr^{n-1} < M$, $\forall n \in \mathbb{N}$. Therefore,

$$|na_n x^{n-1}| = \frac{nr^{n-1}}{\rho} |a_n| \rho^n \leq \frac{M}{\rho} |a_n| \rho^n \quad \forall n \in \mathbb{N}$$

and the series $\sum |a_n| \rho^n$ converges since $\rho < R$ (remember that for all $|x| < R$ the power series converges absolutely).

Finally, the comparison test implies that $\sum na_n x^{n-1}$ converges absolutely.

Conversely, suppose that $|x| > R$, then $\sum |a_n| x^n$ diverges (since $\sum a_n x^n$ diverges) and

$$|na_n x^{n-1}| \geq \frac{1}{|x|} |a_n| x^n \quad \forall n \in \mathbb{N},$$

so the comparison test implies that $\sum na_n x^{n-1}$ diverges.

In conclusion, the power series $g(x) = \sum_{n=1}^{\infty} na_n (x-c)^{n-1}$ has the same radius of convergence $R$ as the power series $f$.\[\square\]

Similarly, using the theorem above and the fundamental theorem of calculus, the following theorem for integration of power series can be proven.

**Theorem 8 (Term-by-term integration of a power series).** Suppose that the power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

has radius of convergence $R$. Then the power series

$$g(x) = \sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1} + C$$

is well defined and it also has radius of convergence $R$.

Moreover, the power series $g$ is one anti-derivative of the power series $f$.\[\square\]
Example 1. Find a power series representation for the function (wherever it makes sense)

$$f(x) = \frac{1}{(1-x)^2}.$$ 

We have that

$$\frac{1}{(1-x)^2} = \left[ \frac{1}{1-x} \right]'$$

and for $|x| < 1$ (geometric series; its radius of convergence is $R = 1$)

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$ 

Therefore, for $|x| < 1$ (the radius of convergence is also equal 1)

$$\frac{1}{(1-x)^2} = \left[ \sum_{n=0}^{\infty} x^n \right]' = \sum_{n=0}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty} (n+1)x^n.$$ 

Example 2. Find a power series representation for the function (wherever it makes sense)

$$f(x) = \arctan(x).$$ 

We know that

$$\arctan(x) = \int \frac{dx}{1+x^2}$$

and for $|x| < 1$ (geometric series; its radius of convergence is $R = 1$)

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$ 

Therefore, for $|x| < 1$ (the radius of convergence is also equal 1)

$$\arctan(x) = \int \sum_{n=0}^{\infty} (-1)^n x^{2n} = C + \sum_{n=0}^{\infty} (-1)^n \int x^{2n} \, dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

and since $\arctan(0) = 0$, then $C = 0$. In conclusion,

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

with radius of convergence equal 1.