Addition and Multiplication Laws (Axioms for \(a + b\) and \(a \cdot b \equiv ab\))

**Definition:** Let \(F\) be a set consisting of at least two elements. Assume for any \(a, b \in F\) there are two operations, \(a + b \in F\) and \(a \cdot b \equiv ab \in F\) defined. Then \(F\) is called a **field**, if for any \(a, b, c \in F\) the following relations hold:

- **A1:** \(a + (b + c) = (a + b) + c \equiv a + b + c\)  
  Associative Addition Law
- **A2:** \(a + b = b + a\)  
  Commutative Addition Law
- **A3:** \(\exists 0 \in F\) such that \(a + 0 = a\ \forall a \in F\)  
  Existence of a Neutral Element for “+”
- **A4:** \(\forall a \exists (-a)\) such that \(a + (-a) = 0\)  
  Existence of Inverse Elements for “+”
- **M1:** \(a(b + c) = (ab)c \equiv abc\)  
  Associative Multiplication Law
- **M2:** \(ab = ba\)  
  Commutative Multiplication Law
- **M3:** \(\exists 1 \in F\) such that \(a \cdot 1 = a\ \forall a \in F\)  
  Existence of a Neutral Element for “.”
- **M4:** \(\forall a \neq 0 \exists a^{-1}\) such that \(aa^{-1} = 1\)  
  Existence of Inverse Elements for “.”
- **DL:** \(a(b + c) = ab + ac\)  
  Distributive Law

**Theorem 3.1:** Let \(F\) be a field with addition “+” and multiplication “.”. Then for any \(a, b, c \in F\) we have the following:

1. If \(a + c = b + c \Rightarrow a = b\)
2. \(a \cdot 0 = 0\)
3. \((-a)b = -(ab)\). In particular: \((-1)a = -(1a) = -a\)
4. \((-a)(-b) = ab\). In particular: \((-1)(-1) = 1 \cdot 1 = 1\)
5. If \(ac = bc\) and \(c \neq 0 \Rightarrow a = b\)
6. If \(ab = 0 \Rightarrow a = 0\) or \(b = 0\)

**Ordering Axioms**

**Definition:** Let \(F\) be a field. \(F\) is called an **ordered field** if there exists a relation \(a \leq b\) between any two elements \(a, b \in F\) with the following properties:

- **O1:** For any \(a, b\) either \(a \leq b\) or \(b \leq a\)
- **O2:** If \(a \leq b\) and \(b \leq a\) \(\Rightarrow a = b\)
- **O3:** If \(a \leq b\) and \(b \leq c\) \(\Rightarrow a \leq c\) (Transitive Law)
- **O4:** If \(a \leq b\) \(\Rightarrow a + c \leq b + c\) for any \(c\)
- **O5:** If \(a \leq b\) and \(0 \leq c\) \(\Rightarrow ac \leq bc\)

**Theorem 3.2:** Let \(F\) be an ordered field. Then we have the following:

1. If \(a \leq b \Rightarrow -b \leq -a\)
2. If \(a \leq b\) and \(c \leq 0 \Rightarrow bc \leq ac\)
3. If \(0 \leq a\) and \(0 \leq b \Rightarrow 0 \leq ab\)
4. \(0 \leq a^2\ \forall a \in F\)
5. \(0 < 1\)
6. If \(0 < a \Rightarrow 0 < a^{-1}\)
7. If \(0 < a < b \Rightarrow 0 < b^{-1} < a^{-1}\)

**Note:** \(a < b\) means \(a \leq b\) and \(a \neq b\) (or NOT \(b \leq a\)).