MATH 345: Sample Final Problems

1: Classify the following differential equations as nonlinear and separable, nonlinear and non-separable, linear homogeneous, or linear inhomogeneous. Note: You don’t need to solve the equations, just classify!

(a) \( t + 3y \frac{dy}{dt} = 4t^2 + \sin(t) \)
(b) \( t \frac{dy}{dt} = 4t^2y + 2t \)
(c) \( \frac{dy}{dt} = \sin(t)ye^{-t^2} \)
(d) \( t + 3y \frac{dy}{dt} = y + 4t^2 + \sin(t) \)

2: A retired person has an amount of \( S(t) \) dollars on a retirement savings account at time \( t \). Interest is compounded continuously. If withdrawals for living expenses are made continuously at a constant rate of \( W \) dollars per year, \( S(t) \) satisfies the initial value problem

\[ S' = rS - W, \quad S(0) = S_0, \]

where \( r \) is the annual interest rate and \( S_0 \) is the initial balance on the account.

(a) Determine \( S(t) \) as function of \( t \).
(b) If the annual interest rate is 6\%, how large must the initial balance \( S_0 \) at least be to allow an annual withdrawal of \( W = $16,000 \) per year for 20 years?

Note: In the final a table for calculating numbers would be provided since calculators are not allowed.

3: In 1997, a certain city had a population of 80,000 people. In 2002 the population was 120,000 people. The city planners did not wish to limit population growth until the population reaches 180,000. Assuming the population \( P(t) \) satisfies the equation (Malthusian model)

\[ \frac{dP}{dt} = kP, \]

when does or did this occur?

Note: In the final a table for calculating numbers would be provided since calculators are not allowed.

4: Solve the initial value problem

\[ y' = y - y^2, \quad y(0) = 1/2. \]

5: Solve the initial value problem

\[ \frac{dy}{dt} = -2ty + e^{-t^2}, \quad y(0) = 1. \]
‡ 6: Find the value of the parameter \( a \) for which the system of equations
\[
\begin{align*}
    x + w &= 1 \\
    y + z &= 1 \\
    x + y + z + w &= a
\end{align*}
\]
has a solution. For that value, compute a parametric representation of the solution set. Also determine a basis of the nullspace of the matrix of coefficients on the left hand sides of the equations.

‡ 7: For which value of the parameter \( a \) are the vectors
\[
\begin{align*}
    \mathbf{v}_1 &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},
    \mathbf{v}_2 &= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},
    \mathbf{v}_3 &= \begin{bmatrix} a \\ 2 \\ 2a \end{bmatrix}
\end{align*}
\]
linearly dependent? For that value find numbers \( c_1, c_2, c_3 \), not all zero, such that
\[
c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3 = \mathbf{0}.
\]

‡ 8: Find a fundamental set of solutions of \( \mathbf{x}' = A \mathbf{x} \) for
\[
A = \begin{bmatrix}
    0 & 1 & 0 \\
    1 & 0 & 1 \\
    0 & 0 & -2
\end{bmatrix}.
\]

‡ 9: Find a fundamental set of real solutions of \( \mathbf{x}' = A \mathbf{x} \) for
\[
A = \begin{bmatrix}
    0 & -1 & 0 \\
    1 & 0 & 1 \\
    0 & 0 & -1
\end{bmatrix}.
\]

‡ 10: Sketch the phase plane portraits of the systems \( \mathbf{x}' = A \mathbf{x} \) for the following matrices \( A \):
\[
\begin{align*}
(a) \quad A &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, &
(b) \quad A &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.
\end{align*}
\]
In the case of distinct real eigenvalues, include all (straight) half line solutions and one generic trajectory in each of the four regions separated by the half line solutions. In case of a node, your sketch should show the tangency of the generic trajectories to the correct half line solutions at the origin. For complex eigenvalues, sketch one generic trajectory in case of a spiral, and three generic trajectories in case of a center. On each trajectory
indicate the direction of motion by an arrow.

‡ 11: Classify the type of phase plane portrait of $\mathbf{x}' = A\mathbf{x}$ for the following matrices $A$. Use one of the following: Spiral or nodal sink or source, saddle, center. Also state whether the origin is a sink, stable but not a sink, a source, unstable but not a source. *Note:* You don’t need to sketch phase portraits - just classify!

\[
A = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}
\]

\[
A = \begin{bmatrix} -1 & -3 \\ 1 & 1 \end{bmatrix}
\]

\[
A = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}
\]

\[
A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}
\]

\[
A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}
\]

‡ 12: Find the solution to the initial value problem

\[
\mathbf{x}' = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} \sin 2t \\ 2 \cos 2t \end{bmatrix}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
\]

‡ 13: Find the solution to the initial value problem

\[
\mathbf{x}' = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}\mathbf{x} + e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
\]

*Hint:* The matrix, $A$, has only one eigenvalue. It is easy to find $e^{At}$ and you can use this as fundamental matrix.

‡ 14: A mass of $1\text{ kg}$ is attached to a spring with spring constant $k = 16\text{ kg/s}^2$. The system is placed in a viscous medium that supplies a damping constant $\mu = 8\text{ kg/s}$. Initially the mass is at rest. Then the mass is given a sharp tarp at its equilibrium position, imparting an instantaneous initial velocity of $1\text{ m/s}$.

(a) Find the position of the mass at time $t$.

(b) For which value of $t$ does the mass reach its maximal position?

‡ 15: A mass of $1\text{ kg}$ is attached to a spring with spring constant $k = 2\text{ kg/s}^2$. The system is placed in a viscous medium that supplies a damping constant $\mu = 2\text{ kg/s}$. Initially the mass is at rest. Then the mass is given a sharp tarp at its equilibrium position, imparting an instantaneous initial velocity of $1\text{ m/s}$. Find the position of the mass at time $t$. 

3
\[ 16: \text{(a) Find the gain } G(\omega) \text{ and the phase angle } \phi(\omega) \text{ of the steady periodic solution, } x_p(t), \text{ for the forced harmonic oscillator equation} \]

\[ x'' + x' + x = A \cos \omega t. \]

\[ \text{(b) Find the frequency } \omega_m \text{ for which the gain is a maximum.} \]

\[ 17: \text{(a) Find the solution of the initial value problem} \]

\[ x'' + 4x' + 4x = 8 \cos 2t, \quad x(0) = 0, \quad x'(0) = 0. \]

\[ \text{(b) Place the steady periodic solution of the equation, } x_p(t), \text{ in the form } x_p(t) = A \cos(2t - \phi), \text{ that is, determine } A \text{ and } \phi. \]

\[ 18: \text{Write down the trial form of the particular solution } y_p(t) \text{ you have to use if you apply the (real) method of undetermined coefficients to the equations below.} \]

\[ \textbf{Note:} \text{ You don’t have to solve for the coefficients!} \]

\[
\text{(a) } y'' + y' + y = \cos t \\
\text{(b) } y'' + 3y' + 2y = e^{-2t} \\
\text{(c) } y'' + y = te^{-t} \sin t \\
\text{(d) } y'' + y = \sin t
\]

\[ 19: \text{Solve the initial value problem} \]

\[ x'' - x = e^{2t}, \quad x(0) = 0, \quad x'(0) = 1 \]

\[ \text{using} \]

\[ \text{(a) the method of undetermined coefficients,} \]

\[ \text{(b) the Laplace transform method.} \]

\[ 20: \text{Solve the initial value problem} \]

\[ x'' + 3x' + 2x = e^{-t}, \quad x(0) = 0, \quad x'(0) = 0 \]

\[ \text{using the Laplace transform method.} \]

\[ 21: \text{Find the solution of the initial value problem} \]

\[ x'' + x = \begin{cases} 1 & \text{if } 0 \leq t < \pi, \\ 0 & \text{if } t \geq \pi \end{cases}, \quad x(0) = 0 = x'(0), \]

\[ \text{for } t \geq 0. \text{ Give a piecewise representation of } x(t) \text{ in the intervals } 0 \leq t < \pi \text{ and } t \geq \pi. \]

\[ \text{You can use any method; recommended is the Laplace transform method.} \]

\[ \textbf{Reminder:} \mathcal{L}\{H(t-c)f(t-c)\}(s) = e^{-sc}F(s) \text{ where } F(s) = \mathcal{L}(f)(s). \]

\[ 22: \text{Find the solution to the initial value problem} \]

\[ x'' + x = \delta(t-2\pi) + \delta(t-4\pi), \quad x(0) = x'(0) = 0, \]

\[ 4 \]
for $t \geq 0$, where $\delta(t)$ is Dirac’s delta function. Give a piecewise definition of the solution in the intervals $0 \leq t < 2\pi$, $2\pi \leq t < 4\pi$ and $t \geq 4\pi$. You can use any method; recommended is the Laplace transform method.

*Reminder:* $\mathcal{L}\{\delta(t - c)\} = e^{-sc}$ and $\mathcal{L}^{-1}\{e^{-sc}F(s)\} = H(t - c)f(t - c)$ if $F(s) = \mathcal{L}(f)(s)$.

\# 23: Solve the initial value problem
\[
y^{(3)} - 2y'' - y' + 2y = 0, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 3.
\]

*Hint:* $\lambda^3 - 2\lambda^2 - \lambda + 2 = (\lambda^2 - 1)(\lambda - 2)$. 
Solutions for Sample Final Problems:

**#1**  
(a) \( t + 3y \frac{dy}{dt} = 4t^2 + \sin t \Rightarrow \frac{dy}{dt} = \frac{1}{3y} [4t^2 + \sin t - t] \)  
nonlinear and separable  
(b) \( t \frac{dy}{dt} = 4t^2 y + 2t \Rightarrow \frac{dy}{dt} = 4ty + 2 \)  
linear inhomogeneous  
(c) \( \frac{dy}{dt} = [e^{-t^2}]y \)  
linear homogeneous  
(d) \( t + 3y \frac{dy}{dt} = y + 4t^2 + \sin t \)  
nonlinear and nonseparable

**#2**  
(a) \( S' = rS - W \); partial sol.: \( S_p = A \) (=const)  
\( \Rightarrow 0 = rA - W \Rightarrow A = \frac{W}{r} \)  
general sol.: \( S(t) = C_1 e^{rt} + \frac{W}{r} \), IC: \( S(0) = C_1 + \frac{W}{r} = S_0 \)  
Solution to IVP: \( S(t) = (S_0 - \frac{W}{r}) e^{rt} + \frac{W}{r} \)  
(b) Let \( S(t^*) = 0 \Rightarrow \frac{W}{r} + (S_0 - \frac{W}{r}) e^{rt^*} = 0 \Rightarrow S_0 = \frac{W}{r} (1 - e^{-rt^*}) \)  
With \( t^* = 20, r = 0.06, W = 16 \cdot 10^3 \):  
\( S_0 = \frac{16 \cdot 10^3}{0.06} (1 - e^{-0.06 \cdot 20}) = \$186,348.21 \)

**#3**  
\( 1997 \rightarrow t = 0 \): \( P_o = 8 \cdot 10^4 \)  
\( 2002 \rightarrow t = 5 \): \( P(5) = 12 \cdot 10^4 \Rightarrow 12 \cdot 10^4 = 8 \cdot 10^4 e^{5k} \Rightarrow k = \frac{\ln 3}{5} \approx 0.082/yr \)  
Then \( P(t^*) = 8 \cdot 10^4 e^{kt^*} = 18 \cdot 10^4 \Rightarrow t^* = \frac{k}{\ln \left(\frac{9}{4}\right)} \approx 10 \)  
\( \Rightarrow 180,000 \) is reached in \( 2007 \)
\[ \frac{dy}{dt} = y - y^2 \rightarrow \int \frac{dy}{y - y^2} = \int dt + C' = t + C' \]

Use \( \frac{1}{y - y^2} = \frac{1}{y} - \frac{1}{y - 1} \Rightarrow \int \frac{dy}{y - y^2} = \ln|y| - \ln|y - 1| = \ln\left|\frac{y}{y - 1}\right| = t + C' \)

Invoke IC: \( t = 0, y = \frac{1}{2} \Rightarrow \ln\left|\frac{1/2}{1-1/2}\right| = \ln(1) = 0 = C' \)

\( \Rightarrow \left|\frac{y}{y - 1}\right| = \frac{y}{1 - y} = e^t \quad (\star: \text{since } y(0) = 1/2) \)

\( \Rightarrow y(t) = \frac{e^t}{1 + e^t} = \frac{1}{1 - e^{-t}} \)

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\[ \#5 \quad \text{Find general solution of } \frac{dy}{dt} = -2ty + e^{-t^2} \]

(a) Int. Factor: \( g = \exp\left(\int 2tdt\right) = e^{t^2} \)

\( \Rightarrow e^{t^2} \left( \frac{dy}{dt} + 2ty \right) = 1 \Rightarrow \frac{d}{dt} \left( e^{t^2} y \right) = 1 \)

\( \Rightarrow e^{t^2} y = t + C' \Rightarrow y(t) = (t + C')e^{-t^2} \)

(b) Var. of Pars.: \( y'_h = -2ty_h \Rightarrow y_h(t) = \exp(-\int 2tdt) = e^{-t^2} \)

Part. sol.: \( y_p(t) = y_h(t) \int \frac{e^{-t^2}}{y_h(t)} \, dt = e^{-t^2} \int dt = te^{-t^2} \)

Gen. sol.: \( y(t) = C'e^{-t^2} + te^{-t^2} \)

Match \( C' \) to IC: \( y(0) = C' = 1 \)

\( \Rightarrow y(t) = (1 + t)e^{-t^2} \) is sol. to IVP
Augmented matrix:

\[ M = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & a \\ 1 & 1 & 1 & a \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & a-1 \\ 0 & 1 & 0 & a-2 \end{bmatrix} \]

Solutions exist if \( a = 2 \). For \( a = 2 \):
\[
\begin{cases} x + w = 1 \\ y + z = 1 \end{cases}
\]
Set \( z = s, \ w = t \) \( \Rightarrow \) \( x = 1 - t, \ y = 1 - s \)
\[
\Rightarrow X = \begin{bmatrix} 1-t \\ 1-s \\ s \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}
\]

basis of null \( (A) \)

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\[ \begin{bmatrix} 1 & 0 & a \\ 1 & 1 & 2 \\ 1 & 1 & 2a \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & a \\ 1 & 1 & 2-a \\ 0 & 1 & a \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & 2-a \\ 0 & 0 & 2a-2 \end{bmatrix} \]

\[ \begin{bmatrix} u_1, u_2, u_3 \end{bmatrix} \]

\[ \text{REF}([u_1, u_2, u_3]) : \]
free variable if \( a = 1 \)
Continue with \( a = 1 \):
\[ [u_1, u_2, u_3] \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0 \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_2 \\ c_3 \end{bmatrix} = 0 \]
Free variable \( c_3 \), set \( c_3 = 1 \) \( \Rightarrow \) \( c_1 = c_2 = -1 \)
\[
\Rightarrow -u_1 - u_2 + u_3 = 0
\]

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\[ A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -2 \\ 0 & 0 & 2 \end{bmatrix}, \ p(\lambda) = \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = -(\lambda+2)(-\lambda-1) = -(\lambda+2)(\lambda^2 - 1) \]
\( \lambda_1 = -2, \lambda_2 = -1, \lambda_3 = 1 \) (evals)
\[ A+2I = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & 2/3 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{evec} \quad u_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \]
\[ A+I = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow u_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} ; A-I = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \]
F.S.S. : \( u_1 e^{-2t}, u_2 e^t, u_3 e^t \)
\[ \Rightarrow u_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]
A = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad p(\lambda) = \begin{vmatrix} -\lambda & -i & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = -(\lambda+i) \begin{vmatrix} -\lambda & -i \\ 1 & -\lambda \end{vmatrix} = -(\lambda+i)(\lambda^2+1)

\Rightarrow \text{evals } \lambda_1 = -1, \lambda_2 = i, \lambda_3 = -i

A + I = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & i/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}

solved: \quad x_1(t) = \mathbf{v}_1 e^{-t}

A - iI = \begin{bmatrix} -i & -1 & 0 \\ 1 & -i & 1 \\ 0 & 0 & -1-i \end{bmatrix} \Rightarrow \begin{bmatrix} -i & -1 & 0 \\ 1 & -i & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow x_2 = i

\Rightarrow \text{evect } \mathbf{v}_2 = \begin{bmatrix} -1 \\ i \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}

sols.: \quad x_2(t) = u \cos t - w \sin t = \begin{bmatrix} -\cos t \\ -\sin t \\ 0 \end{bmatrix}

F.S.S.: \quad x_1(t), x_2(t), x_3(t)

x_3(t) = u \sin t + w \cos t = \begin{bmatrix} -\sin t \\ \cos t \\ 0 \end{bmatrix}

\#10

(a) A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \det A = -1 \Rightarrow \text{saddle}

p(\lambda) = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 \Rightarrow \text{evals } \pm 1

\lambda_1 = 1: A - I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}

\lambda_2 = -1: A + I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}

(b) A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}: \quad T = 0, D = 1 \Rightarrow \text{center}

[0 1] [1] = [0]
[1 0] [0] = [-1] \Rightarrow \text{clockwise dir. of } \mathbb{R}^2.
#11 \[ A = \begin{bmatrix} -1 & 3 \\ -1 & 1 \end{bmatrix} \]: \( T = 0, D = -4 \Rightarrow \text{saddle, unstable but not source} \\
A = \begin{bmatrix} -1 & -3 \\ -1 & 1 \end{bmatrix} \]: \( T = 0, D = 2 > 0 \Rightarrow \text{center, stable but not sink} \\
A = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \]: \( T = -2, D = 2, T^2 - 4D = -4 < 0 \Rightarrow \text{spiral sink, sink} \\
A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \]: \( T = 1, D = 1, T^2 - 4D = -3 < 0 \Rightarrow \text{spiral source, source} \\
A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \]: \( T = 3, D = 1, T^2 - 4D = 5 > 0 \Rightarrow \text{nodal source, source} \\

#12 \[ A = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}, p(\lambda) = |\lambda I - A| = \lambda^2 + 4 = 0 \Rightarrow \text{evecs } \pm 2i \\
A - 2iI = \begin{bmatrix} -2i & 1 \\ 0 & -2i \end{bmatrix} \Rightarrow \text{evec } \psi = \begin{bmatrix} 1 \\ 2i \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2i \end{bmatrix} \\
F.S.S.: X_1(t) = \psi \cos 2t - \omega \sin 2t = \begin{bmatrix} \cos 2t \\ -2 \sin 2t \end{bmatrix} \\
F.M.: X(t) = \begin{bmatrix} \cos 2t & \sin 2t \\ -2 \sin 2t & 2 \cos 2t \end{bmatrix} \\
|X(t)| = 2 \cos^2 2t + 2 \sin^2 2t = 2 \Rightarrow X^{-1}(t) = \frac{1}{2} \begin{bmatrix} 2 \cos 2t & -2 \sin 2t \\ 2 \sin 2t & 2 \cos 2t \end{bmatrix} \\
\text{part. sol.}: \int X^{-1}(t)f(t)dt = \frac{1}{2} \int \begin{bmatrix} 2 \cos 2t & -2 \sin 2t \\ 2 \sin 2t & 2 \cos 2t \end{bmatrix} \begin{bmatrix} \cos 2t \\ -2 \sin 2t \end{bmatrix}dt \\
= \frac{1}{2} \int \begin{bmatrix} 0 \\ 2 \end{bmatrix} dt = \begin{bmatrix} 0 \\ t \end{bmatrix} \\
= \psi \rho(t) = X(t) \int X^{-1}(t)f(t)dt = \begin{bmatrix} \cos 2t & \sin 2t \\ -2 \sin 2t & 2 \cos 2t \end{bmatrix} \begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} \sin 2t \\ 2 \cos 2t \end{bmatrix} \\
\text{gen. sol.}: X(t) = X(t) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \psi \rho(t) \\
\text{Match to IC}: X(0) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow c_1 = 1, c_2 = 0 \\
\text{Sol. to IVP}: X(t) = X(t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \psi \rho(t) = \begin{bmatrix} \cos 2t + t \sin 2t \\ -2 \sin 2t + 2 t \cos 2t \end{bmatrix}
\[ A = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}, \quad p(\lambda) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda(\lambda+2)+1 = (\lambda+1)^2 \]

\( \Rightarrow \) single eval \( \lambda = -1 \)

Set \( B = A + I = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \), then \( e^{At} = e^{-t}(I+tB) = e^{-t}\begin{bmatrix} 1+t & -t \\ t & 1-t \end{bmatrix} \)

part. sol.: \( \int e^{At}f(t)\,dt = \int e^{-t}\begin{bmatrix} 1-t & t \\ -t & 1+t \end{bmatrix} e^{-t}\begin{bmatrix} 1 \\ t \end{bmatrix} \,dt \)

\( = \int \begin{bmatrix} 1-t & t \\ -t & 1+t \end{bmatrix} \begin{bmatrix} 1 \\ t \end{bmatrix} \,dt = \begin{bmatrix} 1 \\ t \end{bmatrix} \)

\( \Rightarrow x_p(t) = e^{At}\begin{bmatrix} t \\ t \end{bmatrix} = e^{-t}\begin{bmatrix} 1+t & -t \\ t & 1-t \end{bmatrix}\begin{bmatrix} t \\ t \end{bmatrix} = e^{-t}\begin{bmatrix} t \\ t \end{bmatrix} \)

gen. sol.: \( x(t) = e^{At}c + x_p(t) \)

Since \( x_p(0) = 0 \Rightarrow c = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \)

\( \Rightarrow \text{Sol. to IVP: } x(t) = e^{-t}\begin{bmatrix} 1+t \\ t \end{bmatrix} + e^{-t}\begin{bmatrix} t \\ t \end{bmatrix} = e^{-t}\begin{bmatrix} 1+2t \\ 2t \end{bmatrix} \)

#14 Newton's law: \( x'' + 8x' + 16x = 0 \), IC: \( x(0) = 0, x'(0) = 1 \)

(a) char. eq.: \( \lambda^2 + 8\lambda + 16 = (\lambda+4)^2 = 0 \Rightarrow \lambda = -4, \text{ mult. } 2 \)

F.S.S.: \( e^{-4t}, te^{-4t} \) \( \Rightarrow \) gen. sol.: \( x(t) = e^{-4t}(c_1 + c_2t) \)

IC: \( x(0) = c_1 = 0, x'(0) = -4c_1 + c_2 = 1 \Rightarrow c_2 = 1 \)

Sol. to IVP: \( x(t) = te^{-4t} \)

(b) max position: \( x'(t) = e^{-4t} - 4te^{-4t} = 0 \Rightarrow t_{max} = \frac{1}{4} \text{ sec} \)

#15 Newton's law: \( x'' + 2x' + 2x = 0 \), IC: \( x(0) = 0, x'(0) = 1 \)

ch. eq.: \( \lambda^2 + 2\lambda + 2 = (\lambda+1)^2 + 1 = 0 \Rightarrow \lambda = -1 \pm i \)

F.S.S.: \( e^{-t}\cos t, e^{-t}\sin t \) \( \Rightarrow \) gen. sol.: \( x(t) = e^{-t}(c_1\cos t + c_2\sin t) \)

IC: \( x(0) = c_1 = 0, x'(0) = -c_1 + c_2 = 1 \Rightarrow c_2 = 1 \)

Sol. to IVP: \( x(t) = e^{-t}\sin t \)
#16  Given \( x'' + dx' + \omega_0^2 x = A \cos \omega t \) \((d, \omega > 0, \omega_0 > 0)\) the steady periodic solution is given by
\[ x_p(t) = A G(\omega) \cos(\omega t - \phi(\omega)) \]
with gain: \( G(\omega) = 1/\sqrt{D(\omega)} \) where \( D(\omega) = (\omega_0^2 - \omega^2)^2 + d^2 \omega^2 \)
phase angle: \( \phi(\omega) = \arccot \left( \frac{\omega_0^2 - \omega^2}{d \omega^2} \right) \), \( 0 < \phi < \pi \).

The max gain occurs when
\[ \frac{dG}{dw} = 0 \Rightarrow \frac{dG}{dw} = \frac{2(\omega_0^2 - \omega^2)(-2\omega) + 2d^2 \omega}{\left( (\omega_0^2 - \omega^2)^2 + d^2 \omega^2 \right)^{3/2}} = 0 \], hence at
\[ w = \omega_m = \sqrt{\frac{\omega_0^2 - d^2/2}{\omega_0^2}} \text{ if } d^2 < 2\omega_0^2 \]
\[ w = \omega_m = \sqrt{1} \text{ if } d^2 > 2\omega_0^2 \]

(a) Here: \( d = 1, \omega_0 = 1 \Rightarrow D(\omega) = (1-\omega^2)^2 + 2\omega^2 = \omega^4 - \omega^2 + 1 \)
\[ \Rightarrow G(\omega) = \frac{1}{\sqrt{\omega^4 - \omega^2 + 1}}, \phi(\omega) = \arccot \left( \frac{\omega_0^2 - \omega^2}{d \omega^2} \right) \]

(b) Here \( d^2 = 1 < 2\omega_0^2 = 2 \Rightarrow \omega_m = \sqrt{1-1/2} = \frac{1}{\sqrt{2}} \text{ (resonant frequency)} \)

#17 \( x'' + 4x' + 4x = 8 \cos 2t \), \( x(0) = 0, x'(0) = 0 \)

(a) Steady periodic sol.: \( x_p(t) = A \cos 2t + B \sin 2t \).
Sub in DDE \( \Rightarrow [(4-2^2)A + 8B] \cos 2t + [(4-2^2)B - 8A] \sin 2t = 8 \cos 2t \)
\[ \Rightarrow B = 1, A = 0 \Rightarrow x_p(t) = \sin 2t \]

hom. eq.: \( x'' + 4x' + 4x = 0 \Rightarrow \lambda^2 + 4\lambda + 4 = (\lambda + 2)^2 = 0 \)
\[ \Rightarrow x_h(t) = e^{-2t} (c_1 + c_2 t) \]

gen. sol.: \( x(t) = e^{-2t} (c_1 + c_2 t) + \sin 2t \)
IC: \( x(0) = c_1 = 0, x'(0) = -2c_1 + c_2 + 2 = 0 \Rightarrow c_2 = -2 \)
Sol. to IVP: \( x(t) = \sin 2t - 2t e^{-t} \)

(b) \( x_p(t) = \sin 2t = \cos(2t - \pi/2) \) \((A = 1, \phi = \pi/2)\)

Note: (cf. #16) \( d = 4, \omega_0^2 = 4, \omega = 2 \Rightarrow \phi = \arccot(0) = \pi/2, G = \frac{1}{\sqrt{64}} = \frac{1}{8} \)
#18  \( a \) \( y'' + y' + y = \cos t \). Damped oscillator \( \Rightarrow \) \( y'' + y' + y = 0 \Rightarrow y_p(t) = A \cos t + B \sin t \)

\( b \) \( y'' + 3y' + 2y = e^{-2t} \). (H) \( y'' + 3y' + 2y = 0 \Rightarrow x^2 + 3x + 2 = (x+1)(x+2) = 0 \Rightarrow e^{-2t} \) is sol. of (H), mult. 1 \( \Rightarrow y_p(t) = Ate^{-2t} \)

\( c \) \( y'' + y = t e^{-t} \sin t \); \( y'' + y = 0 \) has F.S.S. \( \cos t, \sin t \), hence \( y_p(t) = e^{-t} \left[ (A+Bt) \cos t + (C+Dt) \sin t \right] \)

\( d \) \( y'' + y = \sin t \); \( \cos t, \sin t \) are sols. of \( y'' + y = 0 \), hence \( y_p(t) = t \left( A \cos t + B \sin t \right) \)

#19 \( x'' - x = e^{2t}, \; x(0) = 0, \; x'(0) = 1 \)

\( a \) hom. eq.: \( x'' - x = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x = \pm 1 \Rightarrow \) F.S.S. \( e^t, e^{-t} \)

part. sol.: \( e^{2t} \) is not sol. of \( x'' - x = 0 \Rightarrow x_p(t) = Ae^{2t} \)

Sub this in ODE \( \Rightarrow A(4-1)e^{2t} = e^{2t} \Rightarrow A = \frac{1}{3} \)

\( \Rightarrow \) gen. sol.: \( x(t) = c_1 e^t + c_2 e^{-t} + \frac{1}{3} e^{2t} \)

IC: \( x(0) = c_1 + c_2 + \frac{1}{3} = 0 \Rightarrow c_1 + c_2 = -\frac{1}{3} \)

\( x'(0) = c_1 - c_2 + 2/3 = 1 \Rightarrow c_1 - c_2 = 1/3 \)

\( \Rightarrow \) \( c_1 = 0 \), \( c_2 = -2/3 \)

\( \Rightarrow \) Sol. to IVP: \( x(t) = \frac{1}{3}(e^{2t} - e^{-t}) \)

\( b \) \( \mathcal{L}(x'' - x) = (s^2 - 1)x - 1 = \mathcal{L}\{e^{2t}\} = \frac{1}{s-2} \)

\( \Rightarrow \) \( X(s) = \frac{1}{s^2 - 1} + \frac{1}{(s-1)(s-2)} = \frac{s-1}{(s-1)(s-2)} = \frac{s-1}{(s-1)(s+1)(s-2)} = \frac{1}{(s+1)(s-2)} \)

\( \mathcal{L}f: \) \( X(s) = \frac{A}{s+1} + \frac{B}{s-2}, \; A = \frac{1}{s-2} \big|_{s=1} = -\frac{1}{3}, \; B = \frac{1}{s+1} \big|_{s=2} = \frac{1}{3} \)

\( \Rightarrow \) \( X(s) = \frac{1}{3} \left( \frac{1}{s-2} - \frac{1}{s+1} \right) \Rightarrow x(t) = \frac{1}{3}(e^{2t} - e^{-t}) \) as in (a)
20. \(x'' + 3x' + 2x = e^{-t}, \ x(0) = 0 = x'(0)\)

Apply \(\mathcal{L}\) (note IC) \(\Rightarrow (s^2 + 3s + 2)X = \frac{1}{s+1}\) \((\mathcal{L}\{e^{-t}\}(s) = \frac{1}{s+1}\))

\[\Rightarrow X(s) = \frac{1}{(s+1)(s^2 + 3s + 2)} = \frac{1}{(s+1)(s+1)(s+2)} = \frac{1}{(s+1)^2(s+2)}\]

PDF: \(\dot{X}(s) = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s+2}\)

\[A = \frac{d}{ds} \frac{1}{s+2} \bigg|_{s=-1} = -\frac{1}{(s+2)^2} \bigg|_{s=-1} = -1\]

\[B = \frac{1}{s+2} \bigg|_{s=-1} = 1, \ C = \frac{1}{(s+1)^2} \bigg|_{s=-2} = 1\]

\[\Rightarrow X(s) = -\frac{1}{s+1} + \frac{1}{(s+1)^2} + \frac{1}{s+2} \Rightarrow x(t) = -e^{-t} + te^{-t} + e^{-2t}\]

21. \(x'' + x = f(t) = \begin{cases} 1, & 0 \leq t < \pi \\ 0, & t \geq \pi \end{cases}\), \(x(0) = 0 = x'(0)\)

\(f(t) = 1 - H(t-\pi) \Rightarrow \mathcal{L}\{f\}(s) = \frac{1}{s} - \frac{e^{-\pi s}}{s}\)

Apply \(\mathcal{L}\) (note IC) \(\Rightarrow (s^2 + 1)X = \frac{1}{s} (1 - e^{-\pi s})\)

Use \(\frac{1}{s(s^2+1)} = \frac{1}{(s-i)(s+i)s} = \frac{A}{s} + \frac{B}{s-i} + \frac{C}{s+i}\) (PDF)

\[A = \frac{1}{s^2+1} \bigg|_{s=0} = 1, \ B = \frac{1}{s(s+i)} \bigg|_{s=i} = -\frac{1}{2}\]

\[\Rightarrow \frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{1}{2} \left( \frac{1}{s-i} + \frac{1}{s+i} \right) = \frac{1}{s} - \frac{1}{s^2+1}\]

[Simpler method: \(\frac{1}{s(s^2+1)} = s \frac{1}{s^2(s+1)} = s \left( \frac{1}{s^2} - \frac{1}{s+1} \right) = \frac{1}{s} - \frac{s}{s^2+1}\)]

\[\Rightarrow X(s) = \frac{1}{s} - \frac{s}{s^2+1} - e^{-\pi s} \left( \frac{1}{s} - \frac{1}{s^2+1} \right)\]

\(\Rightarrow x(t) = 1 - \cos t - H(t-\pi)[1 - \cos(t-\pi)] = \begin{cases} 1 - \cos t, & 0 \leq t < \pi \\ -\cos t + \cos(t-\pi), & t \geq \pi \end{cases}\)
#22 \[ x'' + x = \delta(t-2\pi) + \delta(t-4\pi), \ x(0) = 0 = x'(0) \]
apply \( L \) (note IC) \( \Rightarrow \) \( (s^2 + 1)X = e^{-2\pi s} + e^{-4\pi s} \)
\( \Rightarrow \) \( X(s) = \frac{e^{-2\pi s}}{s^2 + 1} + \frac{e^{-4\pi s}}{s^2 + 1} \)
\( \Rightarrow \) \( X(t) = \text{H}(t-2\pi) \frac{\sin(t-2\pi)}{\sin t} + \text{H}(t-4\pi) \frac{\sin(t-4\pi)}{\sin t} \)
\( = \left[ \text{H}(t-2\pi) + \text{H}(t-4\pi) \right] \frac{\sin t}{\sin t} \)
\( = \begin{cases} 
0, & 0 \leq t < 2\pi \\
\sin t, & 2\pi \leq t < 4\pi \\
2\sin t, & t \geq 4\pi 
\end{cases} \)

#23 \[ y''' - 2y'' - y' + 2y = 0, \ y(0) = 0 = y'(0), y''(0) = 3 \]
ch. eq.: \[ x^3 - 2x^2 - x + 2 = (x-1)(x-2) = 0 \]
\( \Rightarrow \) F.S.S. \( e^{-t}, e^t, e^{2t} \)
\( \Rightarrow \) gen. sol.: \( x(t) = c_1 e^{-t} + c_2 e^t + c_3 e^{2t} \)
IC: \( x(0) = c_1 + c_2 + c_3 = 0 \)
\( x'(0) = -c_1 + c_2 + 2c_3 = 0 \)
\( x''(0) = c_1 + c_2 + 4c_3 = 3 \)
\( \Rightarrow \) \( \begin{cases} 
-1 = c_1 + c_2 \\
-2 = c_1 + c_2 
\end{cases} \)
\( \Rightarrow \) \( \begin{cases} 
c_2 = -3/2 \\
c_1 = 1/2 
\end{cases} \)
\( \Rightarrow \) Sol. to IVP: \( x(t) = \frac{1}{2} e^{-t} - \frac{3}{2} e^t + e^{2t} \)