Introduction

Functions

A function is a mathematical rule that assigns a value to every element in a set of objects. The set of “inputs” of the function is called the domain of the function, while the set which contains the outputs of the function is called the codomain. Most of the functions we will encounter in this course will assign a real number to another real number, but there are many other functions. For example, suppose you are at a deli. The menu looks something like this:

<table>
<thead>
<tr>
<th>Sandwhich:</th>
<th>Price:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ham</td>
<td>3.00</td>
</tr>
<tr>
<td>Tuna</td>
<td>2.75</td>
</tr>
<tr>
<td>Roast Beef</td>
<td>3.50</td>
</tr>
<tr>
<td>Egg Salad</td>
<td>3.00</td>
</tr>
</tbody>
</table>

It may not be immediately apparent, but this menu is actually a function. It takes an element from the set \{Ham, Tuna, Roast Beef, Egg Salad\} and assigns it a value in the set \{2.75, 3.00, 3.50\}.

Now suppose we want to create another function by taking each price in the set \{2.75, 3.00, 3.50\} and using the above menu to assign it a value in the set \{Ham, Tuna, Roast Beef, Egg Salad\}. We quickly run into an issue: what value does the price 3.00 assume? When we define a function, we want each element in the domain to be assigned at most one value in the range. According to the menu, the function could assign 3.00 to either “Ham” or “Tuna”, so there is ambiguity in what the value of 3.00 should be, and thus we cannot use the menu to define a function in this way.

Now let’s quickly review the types of functions we will study in this course. The domain and range of all functions we consider will be subsets of the real numbers \(\mathbb{R}\) (these are numbers that you are familiar with from previous math courses, such as \(\sqrt{2}, \pi, e, \) and \(\frac{3}{4}\)). If \(A\) is the domain of a function \(f\) and \(B\) is the codomain of \(f\), then we will write \(f : A \rightarrow B\) as shorthand for “\(f\) is a function that maps \(A\) to \(B\)”. The range of a function \(f\) is the set of values in \(B\) that are actually mapped to by \(f\), and is denoted \(f(A)\). To illustrate the
difference between the codomain and range, consider the function $f : \mathbb{R} \to \mathbb{R}$, defined by the rule $f(x) = x^2$. A graph is shown in figure 1.

The domain of $f$ is $\mathbb{R}$, as we only expect real numbers as inputs. The codomain is also $\mathbb{R}$, as we only expect $f$ to assume values which are real numbers. The range, however, is the set of non-negative real numbers, which we will use interval notation to denote as $[0, \infty)$.

Now, suppose we define another function $g : \mathbb{R} \to [0, \infty)$ by the rule $g(x) = x^2$. This function has the exact same graph as $f$ defined as above, but is not considered the same function as $f$ because it has a different codomain. In general, we say two functions are equal if they have the same domain and codomain, and their values agree at on every element of the domain. Let’s phrase this another way in a definition:

**Definition (Equality of functions).** Two real valued functions $F : A \to B$ and $G : A \to B$ are equal if for every real number $a$ in the set $A$, we have $F(a) = G(a)$.

As another example, define the functions $f : \mathbb{R} \to [-1, 1]$ and $g : \mathbb{R} \to [-1, 1]$ by $f(x) = \sin x$ and $g(x) = \sin(x + 2\pi)$. While these functions are defined by different formulas, this difference is superficial, and the functions are equal according to our above definition. To check this, first note the the domain of both functions is $\mathbb{R}$, then observe that the range of both functions is $[-1, 1]$. Finally, we recall a trig identity and note that for any real number $x$, $\sin(x) = \sin(x + 2\pi)$. Thus the two functions are equal. The common graph of both functions is shown in figure 2.

The reason for being so fussy about the domain and codomain of our functions becomes more apparent when we try to define inverse functions. For instance, suppose we are trying to define a square root function. We first must ask ourselves what a square root is. This is accomplished as follows:

**Definition (Square roots).** A square root of a real number $a$ is a real number $b$ with the property that $b^2 = a$. 

Figure 1:
We encounter two problems as we try to define a function that takes real numbers as inputs and returns their real square roots. First, not every real number has a square root in the real numbers. The square roots of negative numbers are complex valued, and we will not want to consider these numbers in this course. Thus, the first step in defining a square root function will be to stipulate that our domain is $[0, \infty)$. The second problem is that positive real numbers do not have unique square roots by our definition. For instance, the numbers 1 and $-1$ are both square roots of 1, because $1^2 = 1$ and $(-1)^2 = 1$. Luckily, we can avoid this ambiguity by defining the square root function to only return the positive square root of its input. Our definition is thus as follows:

**Definition (Square root function).** The square root function is the function $f : [0, \infty) \to \mathbb{R}$ which sends $x$ to its positive square root. We write $f(x) = \sqrt{x}$.

Here is a list of families of functions whose properties you should review for this class.

1. Polynomials
   - (a) Lines
   - (b) Quadratics
   - (c) Higher Order polynomials

2. Power functions

3. Rational functions

4. Trig Functions
   - (a) Special Right triangles and SOH, CAH, TOA
   - (b) Identities: Pythagorean identities, double angle formula, etc.
What is calculus?

Informally, calculus is the art of approximating quantities that we could not otherwise measure. There are two primary branches of calculus, called differential calculus and integral calculus. Differential calculus is concerned with approximating functions with lines. The motivation for doing so is that lines are among the easiest functions to work with. This approximation can only be accomplished locally, that is, really close to some point in the domain of the function. For example, consider the function $f : [0, 1] \rightarrow \mathbb{R}$ defined by $f(x) = x^5$. It’s graph is shown in figure 3.

Now we are going to zoom in on the function about $x = 0.7$. Note the axis values in figure 4.

When viewed in this window, the function looked less like a curve and more like a line. Zooming in even further, as in figure 5, it is hard to distinguish the function from a line.

As we zoom in further and further, the slopes of the lines we see will approach some value. We call this “ideal” slope of the function at a point the derivative of the function. It is good to be aware that not every function can be nicely approximated by lines in this manner. Those that can are called differentiable.

Integral calculus is concerned with approximating areas. For instance, we may want to know the area of region enclosed between the the graph of a function and the $x$ axis. We do this by breaking the area we want to approximate into smaller areas which are easier to...
compute. The standard approach is to use rectangles. As the number of rectangles we use in an approximation increases, the total area of the rectangles gets closer and closer to the area under the curve.

Both branches of calculus rely on the concept of a limit. A limit gives us a way to “zoom in closer and closer” in differential calculus and ”subdivide into infinitessimally small areas” in the case of integral calculus. These will be defined rigorously in chapter 2.

Chapter 2

Section 2.3

Logic and Quantifiers

Much of mathematics is concerned with determining the validity of statements about mathematical objects. In order to do this, we rely on the foundations of logic. Any statement that we make, inside or outside of mathematics, is either true or false. For instance, I could say, “the grass is green”, which is a true statement, or “the sky is green” which is (hopefully) a false statement. Often, we abbreviate statements with predicates, such as \( P \), \( Q \), or \( R \), and truth values with the letters \( T \) for “true” and \( F \) for “false”. For instance, I could assign ”the grass is green” the predicate \( P \), and this predicate would assume the truth value \( T \).

There are a variety of operations that we can perform on predicates. These return new predicates with different truth values. To display the effects of these operations, we use truth tables. The first operation we consider is negation. If \( P \) is a predicate, then the negation of \( P \) is the predicate \( \neg P \) (read “not \( P \)”) which is true when \( P \) is false and false when \( P \) is true. This is summarized in the following truth table:

<table>
<thead>
<tr>
<th>( P )</th>
<th>( \neg P )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

The next logical operations are binary operations, meaning that they take two inputs and produce one output. The first is conjunction. The conjunction of predicates \( P \) and \( Q \) is true when \( P \) and \( Q \) are both true, and is false otherwise. This is denoted \( P \land Q \), and is read “\( P \) and \( Q \)”.

The next operation is disjunction. The disjunction of \( P \) and \( Q \) is false when both \( P \) and \( Q \) are false, and is true otherwise. It is denoted \( P \lor Q \), and is pronounced “\( P \) or \( Q \)”.

The truth tables for conjunction and disjunction are shown below.

<table>
<thead>
<tr>
<th>( P )</th>
<th>( Q )</th>
<th>( P \land Q )</th>
<th>( P \lor Q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

The next two logical operations we will encounter are built from negation, conjunction, and disjunction. The first is implication, which models the statement “if \( P \), then \( Q \)”. We may alternatively write this as \( P \implies Q \), pronounced “\( P \) implies \( Q \)”.

In terms of negation and
disjunction, \( P \Rightarrow Q \) means the same as \( \neg P \lor Q \). Next is logical equivalence, which is denoted \( P \Leftrightarrow Q \), and is pronounced “\( P \) if and only if \( Q \)”. In terms of conjunction and implication, \( P \Leftrightarrow Q \) is the same as \( P \Rightarrow Q \land Q \Rightarrow P \). The truth tables for implication and logical equivalence are shown below:

\[
\begin{array}{c|c|c|c}
P & Q & P \Rightarrow Q & P \Leftrightarrow Q \\
\hline
T & T & T & T \\
T & F & F & F \\
F & T & T & F \\
F & F & T & T \\
\end{array}
\]

Let’s see an example of a mathematical statement which we can prove using this logical framework.

**Theorem**. If two functions are equal, then they have the same graph.

**Proof.** To make this rigorous, we need a precise notion of what the graph of a function really is. Suppose \( f : \mathbb{R} \to \mathbb{R} \). The **graph** of \( f \) is the set \( \Gamma_f \) of points \((x, y)\) in the euclidean plane (also denoted \( \mathbb{R}^2 \)) such that \( y = f(x) \). We can show this with **set notation** as

\[
\Gamma_f = \{(x, y) \in \mathbb{R}^2 \mid y = f(x)\}.
\]

We write \( a \in A \) to denote that an object \( a \) is an element of a set \( A \) (or that it is contained in \( A \)). Two sets \( A \) and \( B \) are equal if \( x \in A \Leftrightarrow x \in B \). Now let \( f : \mathbb{R} \to \mathbb{R} \) and \( g : \mathbb{R} \to \mathbb{R} \) be equal as functions, so that \( f(x) = g(x) \) for every \( x \in \mathbb{R} \). We want to show that \( \Gamma_f = \Gamma_g \) as sets. We have the following equivalences:

\[
(x, y) \in \Gamma_f \Leftrightarrow y = f(x) \Leftrightarrow y = g(x) \Leftrightarrow (x, y) \in \Gamma_g.
\]

Thus \( \Gamma_f = \Gamma_g \), so \( f \) and \( g \) have the same graph. \( \square \)

**Exercise**. Determine whether the following statement is true or false:

If two functions \( f \) and \( g \) have the same graph, then \( f = g \).

Next we turn to quantifiers, which are used to build statements. For instance, we may want to make a sweeping generalization about grass, and instead of saying “the grass outside is green”, say “emphall grass is green”. To do this, we need to reference some, perhaps theoretical, collection which contains all the species of grass. We will call this collection \( G \). To refer to a particular species of grass in this collection, we will use a lowercase letter, such as \( g \) or \( h \). Finally, we will let the predicate \( P(g) \) represent the statement “species \( g \) of grass is green”. For instance, if \( g \) were the species “Kentucky Bluegrass”, then the predicate \( P(g) \) represents the statement “Kentucky Bluegrass is green”. If \( h \) is the species “Bermudagrass”, then \( P(h) \) represents the statement “Bermudagrass is green”. We represent the statement “All species of grass are green” with the symbols: \( \forall g \in G, P(g) \). This translates to: “for every species of grass \( g \), \( g \) is green”.

Now we turn to the question of determining whether the statement \( \forall g \in G, P(g) \) is true or
false. In order to show that this statement is true, we need either go through a list of all the possible species of grass and check that every one is green, or we need to construct an argument in which we consider an arbitrary species of grass and then show that the species of grass is green. To show that this statement is false is a simpler task, for we would only need to find one species of grass which is not green. This is called finding a counterexample.

The second quantifier is the existence quantifier. Suppose we wanted to make a weaker statement than “all species of grass are green”, and instead we claim that “there exists a species of grass which is green”. In symbols, this is denoted $\exists g \in G, P(g)$. To show that this statement is true, we only have to find one species of grass that is green. To show that it is false, we need to instead show that every species of grass is not green.

It is possible to string multiple quantifiers together, and in fact, most statements consist of not one but two, three or more quantifiers. For instance, suppose $A$ is the set of all animals, and $Q(a, g)$ is the predicate “animal $a$ eats species of grass $g$”. There are very different statements we can make using this predicate, depending on the quantifiers we use and in what order we use them. The statement

$$\forall g \in G, \exists a \in A, Q(a, g)$$

translates to “for every species of grass $g$, there exists an animal $a$ which eats $g$. What happens if we switch the order of the quantifiers? The statement

$$\exists a \in A, \forall g \in G, Q(a, g)$$

instead translates to “there exists an animal which eats every species of grass”. Think about how you would go about verifying the validity of each of these statements. To prove that statement (1) is true, you would have to first consider an arbitrary species of grass $g$, and then show that there is at least one animal $a$ which eats $g$. Note that the animal $a$ is dependent on the species of grass. To prove statement (2) is true, we would have to come up with one species of grass $g$ so that, no matter which animal we consider, this animal eats $g$. In this case the animal $a$ is said to be independent of the grass $g$.

Negating a statement with quantifiers is easy: you simply switch the quantifiers and then negate the predicate. For example, the negation of statement (1) above is:

$$\neg(\forall g \in G, \exists a \in A, Q(a, g)) = \exists g \in G, \forall a \in A, \neg Q(a, g),$$

which translates to: there exists a grass $g$ which is not eaten by any animal $a$. The negation of statement (2) is

$$\neg(\exists a \in A, \forall g \in G, Q(a, g)) = \forall a \in A, \exists g \in G, \neg Q(a, g).$$

This translates to “for every animal $a$, there exists a grass $g$ which $a$ does not eat.

**Absolute Value and distances**

The formal definition of a limit relies on a notion of distance between two real numbers. Our primary tool for measuring distance will be the absolute value function. This is a piecewise
function $|*| : \mathbb{R} \to \mathbb{R}$ defined by

$$|x| = \begin{cases} 
  x, & \text{if } x \geq 0 \\
  -x, & \text{if } x < 0
\end{cases}$$

The graph of the absolute value function is shown in figure 6.

Note that the absolute value function is non-negative, making it an ideal choice for the distance between two numbers, as it makes no sense for a distance to be negative. Important properties of the absolute value function are its relationship with addition and multiplication.

**Theorem (Properties of the absolute value function).** Given any real numbers $a$ and $b$, we have

1. $|a \cdot b| = |a| \cdot |b|$
2. $|a - b| = |b - a|$
3. $|a + b| \leq |a| + |b|$
4. $|a - b| \geq |a| - |b|$

Items 3 and 4 in the above theorem are called the triangle inequalities, and are crucial in estimating distances.

We now define the distance between two numbers $a$ the absolute value of their difference.

**Definition (distance between real numbers).** The distance between two real numbers $a$ and $b$ is the quantity $|a - b|$.

We can practice with the absolute value function by doing so called “tolerance” problems. These are the focus of lab 1.
Exercise. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 3x - 5$. How close must $x$ be to 2 to ensure that $f(x)$ is within $\frac{1}{10}$ of 1?

Solution: Suppose that the distance between $f(x)$ and 1 is less than one tenth. By the definition of distance, this is the same as saying $|f(x) - 1| < \frac{1}{10}$. Our goal is to deduce some information about the distance between $x$ and 2. The following are logically equivalent:

$$
|f(x) - 1| < \frac{1}{10}
$$

$$
|(3x - 5) - 1| < \frac{1}{10}
$$

$$
|3x - 6| < \frac{1}{10}
$$

$$
3|x - 2| < \frac{1}{10}
$$

$$
|x - 2| < \frac{1}{30}
$$

Thus if $|x - 2| < \frac{1}{30}$, it follows that $|f(x) - 1| < \frac{1}{10}$.

Follow ups:

1. How close must $x$ be to 2 to ensure that $f(x)$ is within $\frac{1}{100}$ of 1?

2. Let $\epsilon > 0$ be an arbitrary positive real number. How close must $x$ be to 2 to ensure that $f(x)$ is within $\epsilon$ of 1?

Exercise. Define $g : [2, \infty) \to \mathbb{R}$ by $g(x) = \sqrt{x - 2}$. How close must $x$ be to 3 to ensure that $g(x)$ is within $\frac{1}{10}$ of 1?

Solution: First assume that $|g(x) - 1| < \frac{1}{10}$. We want to deduce some information about the value of $|x - 3|$. Note that the following statements are equivalent:

$$
|g(x) - 1| < \frac{1}{10}
$$

$$
|\sqrt{x - 2} - 1| < \frac{1}{10}
$$

$$
-\frac{1}{10} < \sqrt{x - 2} - 1 < \frac{1}{10}
$$

$$
\frac{9}{10} < \sqrt{x - 2} < \frac{11}{10}
$$

$$
\frac{81}{100} < x - 2 < \frac{121}{100}
$$

$$
\frac{19}{100} < x - 3 < \frac{21}{100}.
$$
Figure 7: graphs of $f(x) = x^2$, $y = 3$, $y = 3 + \frac{1}{10}$, and $y = 3 - \frac{1}{10}$

Now, if $|x - 3| < \frac{19}{100}$, then $-\frac{19}{100} < x - 3 < \frac{19}{100}$. From here we have $-\frac{19}{100} < x - 3 < \frac{21}{100}$ because $\frac{19}{100} < \frac{21}{100}$. This implies, by the equivalences above, that $|g(x) - 1| < \frac{1}{10}$.

**Exercise.** Define $h : [-1, \infty) \to \mathbb{R}$ by $h(x) = 2\sqrt{x + 1}$. How close must $x$ be to 3 to ensure that $h(x)$ is within 4 of $\frac{1}{5}$?

**Solution:** If $|x - 3| < 4 - \left(\frac{19}{10}\right)^2$, then $|h(x) - 4| < \frac{1}{5}$.

**Exercise.** Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. How close must $x$ be to $\sqrt{3}$ to ensure that $f(x)$ is within $\frac{1}{10}$ of 3?

**Geometric Approach** We want $|x^2 - 3| < \frac{1}{10}$. This is a condition on the distance between the $y$ coordinates of the points on the graph of $f(x) = x^2$ and those of the line $y = 3$. The $y$ coordinates on the graph of $f$ must be between the lines $y = 3 - \frac{1}{10} = \frac{29}{10}$ and $y = 3 + \frac{1}{10} = \frac{31}{10}$ (see figure 7 for a picture). Using this information, we want to deduce some equivalent statement about the $x$ coordinates of these points. Note that $f$ is strictly increasing on $[0, \infty)$. This means that the $x$ coordinates we are looking for are between the $x$-values of the points where $x \geq 0$ and $f$ intersects the lines $y = \frac{29}{10}$ and $y = \frac{31}{10}$. The points of intersection can be found by solving the equations $x^2 = \frac{29}{10}$ and $x^2 = \frac{31}{10}$. The positive solutions are $x = \sqrt{\frac{29}{10}}$ and $x = \sqrt{\frac{31}{10}}$. We have thus determined that $|x^2 - 3| < \frac{1}{10}$ if and only if $\sqrt{\frac{29}{10}} < x < \sqrt{\frac{31}{10}}$.

Now our task is to find some condition on $|x - \sqrt{3}|$ which implies that $\sqrt{\frac{29}{10}} < x < \sqrt{\frac{31}{10}}$. It helps now to draw the interval $(\sqrt{\frac{29}{10}}, \sqrt{\frac{31}{10}})$, along with the point $\sqrt{3}$, which is contained in this interval. This picture should make it clear that if $|x - \sqrt{3}| < \min\{\sqrt{3} - \sqrt{\frac{29}{10}}, \sqrt{3} - \sqrt{\frac{31}{10}}\}$, then $\sqrt{\frac{29}{10}} < x < \sqrt{\frac{31}{10}}$. Using a calculator for approximations, we find that $\min\{\sqrt{3} - \sqrt{\frac{29}{10}}, \sqrt{3} - \sqrt{\frac{31}{10}}\} = \sqrt{\frac{31}{10}} - \sqrt{3}$. Our final conclusion
is: “if $|x - \sqrt{3}| < \sqrt{\frac{31}{10}} - \sqrt{3}$, then $|x^2 - 3| < \frac{1}{10}$”.

**Algebraic Approach (another solution)** We want to eventually deduce some kind of condition on $|x - \sqrt{3}|$. The trick here is to assume that $|x - \sqrt{3}| < \delta$, where we leave $\delta > 0$ unspecified until we know what it should be to make $|x^2 - 3| < \frac{1}{10}$. We now create a string of inequalities that will eventually show that, once we have figured out the necessary assumptions on $\delta$, we are guaranteed that $|x^2 - 3| < \frac{1}{10}$. Notice that, by the difference of squares formula, $|x^2 - 3| = |x + \sqrt{3}||x - \sqrt{3}|$. Since $|x - \sqrt{3}| < \delta$, it follows that $|x + \sqrt{3}| < 2\sqrt{3} + \delta$ (draw a number line to verify this). If we assume that $\delta < \sqrt{3}$, then $|x + \sqrt{3}| < 3\sqrt{3}$. Thus,

$$|x^2 - 3| = |x + 3||x - 3| < 3\sqrt{3}|x - 3| < 3\sqrt{3}\delta.$$

If we further assume that $\delta < \frac{1}{30\sqrt{3}}$, then

$$|x^2 - 3| < 3\sqrt{3}\delta < 3\sqrt{3}\frac{1}{30\sqrt{3}} = \frac{1}{10}.$$

Along the way, we assumed that $\delta < \sqrt{3}$ and that $\delta < \frac{1}{30\sqrt{3}}$. Since $\frac{1}{30\sqrt{3}}$ is the smaller of these two numbers, it is sufficient to assume that $|x - \sqrt{3}| < \frac{1}{30\sqrt{3}}$ to ensure that $|x^2 - 3| < \frac{1}{10}$.

Note that we arrived at two different solutions using these two methods. This is okay, as either condition on $|x - \sqrt{3}|$ implies that $|x^2 - 3| < \frac{1}{10}$.

**The formal definition of a limit**

**Definition (limit).** We say that a real number $\ell$ is the limit of a function $f : \mathbb{R} \to \mathbb{R}$ as $x$ approaches $a$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $0 < |x - a| < \delta$ implies that $|f(x) - f(a)| < \epsilon$. We then write

$$\lim_{x \to a} f(x) = \ell.$$

Translated into logic/quantifier notation, this same definition reads:

**Definition (limit, shorthand notation).** We say that a real number $\ell$ is the limit of a function $f : \mathbb{R} \to \mathbb{R}$ as $x$ approaches $a$ if

$$\forall \epsilon > 0, \exists \delta > 0, 0 < |x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon.$$

**Exercise .** Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 2x, & \text{if } x \neq 1 \\ 0, & \text{if } x = 1 \end{cases}$$

What is $\lim_{x \to 1} f(x)$? Prove that your answer is correct using the definition of a limit.

**Solution:** Consider the graph of the function $f$. There is a hole at the point $(1, 2)$. Nonetheless, the value $f$ approaches 2 as $x$ approaches 1, so it is reasonable to conjecture that
\[ \lim_{x \to 1} f(x) = 2. \] Now we prove this. Let \( \epsilon > 0 \) be arbitrary. We must determine some number \( \delta > 0 \) such that \( 0 < |x - 1| < \delta \) implies \( |f(x) - 2| < \epsilon \). Let's leave the condition on \( \delta \) undetermined for the minute, and try to get an upper bound on \( |f(x) - 2| \) in terms of \( \delta \). Since we know that \( |x - 1| < \delta \), we try to rewrite \( |f(x) - 2| \) in terms of \( |x - 1| \):

\[
|f(x) - 2| = |2x - 2| < 2|x - 1| < 2\delta.
\]

Thus we must choose \( \delta \) so that \( 2\delta < \epsilon \), or equivalently \( \delta < \frac{\epsilon}{2} \).

**Exercise**. Define \( g : \mathbb{R} \to \mathbb{R} \) by

\[
g(x) = \begin{cases} 
\frac{1}{3}x - 2, & x \neq 6 \\
1, & x = 6
\end{cases}
\]

What is \( \lim_{x \to 1} g(x) \)? Prove that your answer is correct using the definition of a limit.

**Solution:** Based on a graph of \( g \), we conjecture that \( \lim_{x \to 6} g(x) = 0 \). Next, we prove it. Let \( \epsilon > 0 \) be arbitrary. We need to determine a condition on \( \delta > 0 \) such that \( 0 < |x - 6| < \delta \) implies \( |g(x) - 0| < \epsilon \). Taking a similar approach to the previous example, we note that if \( 0 < |x - 6| < \delta \), then

\[
|g(x) - 0| = \left| \frac{1}{3}x - 2 \right| = \frac{1}{3} |x - 6| < \frac{\delta}{3}.
\]

Thus if we choose \( \delta < 3\epsilon \), then \( |g(x) - 0| < \epsilon \).

**Exercise**. Define \( h : \mathbb{R} \to \mathbb{R} \) by \( h(x) = \sqrt{x} \). What is \( \lim_{x \to 9} h(x) \)? Prove that your answer is correct using the definition of a limit.

Based on the graph of \( h \) we conjecture that \( \lim_{x \to 9} h(x) = 3 \). To prove it, first assume that \( \epsilon > 0 \). We must find \( \delta > 0 \) so that \( 0 < |x - 9| < \delta \) implies \( |h(x) - 3| < \epsilon \). First we make use of the difference of squares formula: \( a^2 - b^2 = (a - b)(a + b) \) with \( a = x \) and \( b = 9 \) to conclude that

\[
|h(x) - 3| = \left| \frac{x - 9}{\sqrt{x} + 3} \right| < \frac{\delta}{\sqrt{x} + 3}.
\]

Next, we observe that since \( \sqrt{x} + 3 \geq 3 \), it follows that \( \frac{\delta}{\sqrt{x} + 3} \leq \frac{\delta}{3} \), and thus

\[
|h(x) - 3| < \frac{\delta}{\sqrt{x} + 3} \leq \frac{\delta}{3}.
\]

Thus if \( \delta < 3\epsilon \), it follows that \( |\sqrt{x} - 3| < \epsilon \).

**Exercise**. **Question:** If \( f : \mathbb{R} \to \mathbb{R} \) satisfies \( f(a) = \ell \), then is it true that \( \lim_{x \to a} f(x) = \ell \)?

**Answer:** This is not true in general. Consider the function \( f : \mathbb{R} \to \mathbb{R} \) given by

\[
f(x) = \begin{cases} 
1, & x \geq 0 \\
-1 & x < 0
\end{cases}
\]
We show that \( \lim_{x \to a} f(x) \) does not exist by showing that for any \( \ell \in \mathbb{R} \), \( \lim_{x \to a} f(x) \neq \ell \). We must consider two cases: when \( \ell \geq 0 \) and when \( \ell < 0 \). First suppose \( \ell < 0 \). Let \( \epsilon_0 = 1 \). No matter what \( \delta > 0 \) we choose, we can find some \( x_\delta \in (0, \delta) \). Now observe that \( |x_\delta - 0| < \delta \), but

\[
|f(x_\delta) - \ell| = |1 - \ell| \geq |1| - |\ell| = 1,
\]

and therefore \( \lim_{x \to a} f(x) \) cannot be less than zero. Now suppose \( \ell \geq 0 \), and let \( \epsilon_0 = 1 \). No matter what \( \delta > 0 \) we choose, we can find some \( x'_\delta \in (-\delta, 0) \). We now have that \( |x'_\delta - 0| < \delta \), but

\[
|f(x'_\delta) - \ell| = |1 - 1 - \ell| = 1 + \ell \geq 1.
\]

Thus \( \lim_{x \to a} f(x) \) cannot be greater than or equal to zero, and thus cannot be a real number.

**Exercise.** Show that \( \lim_{x \to -2} x^2 = 4 \) using the definition of a limit.

**Solution:** Let \( \epsilon > 0 \) be arbitrary. Choose \( \delta > 0 \) satisfying \( \delta < \min\{1, \frac{\epsilon}{5}\} \). If \( 0 < |x + 2| < \delta \), then \( |x - 2| < \delta + 4 \), and therefore

\[
|x^2 - 4| = |x + 2||x - 2| < \delta|x - 2| < \delta^2 + 4\delta < 5\delta < \epsilon.
\]

(Note that we have used the fact that, if \( 0 < \delta < 1 \), then \( \delta^2 < \delta \). This can be determined by looking at the graphs of \( f(x) = x \) and \( g(x) = x^2 \) on the interval \((0, 1)\).

**Section 2.2**

In the previous section, we gave a precise mathematical definition for the limit of a function. At times, it is rather cumbersome to directly compute limits using this definition. In practice, there are tricks to figure out whether a limit exists and, if so, what it’s value is, which avoid using the original definition. These tricks are the focus of this section.

**Common situations where a limit does exist**

1. Continuous functions: Informally, these are functions whose graphs can be sketched without lifting your pencil from the paper. They have no “jumps”, “holes”, or vertical asymptotes. We can evaluate their limits just by evaluating the function at the point which \( x \) is approaching. Some examples:

   (a) \( \lim_{x \to -2} x^3 = (-2)^3 = -8 \)

   (b) \( \lim_{x \to 8} \sqrt[3]{x} = \sqrt[3]{8} = 2 \)

   (c) \( \lim_{x \to \pi/4} \sin(x) = \sin \left( \frac{\pi}{4} \right) = \frac{\sqrt{2}}{2} \)

   (d) \( \lim_{x \to \log(2)} e^x = e^{\log(2)} = 2 \) (note: I will always use “\( \log(a) \)” to denote the natural logarithm of \( a \), and *not* the logarithm of \( a \) base 10)
\[ \lim_{x \to \pi/6} \cos(x) = \cos \left( \frac{\pi}{6} \right) = \frac{\sqrt{3}}{2}. \]

2. Functions with removable discontinuities: These are functions which are “almost continuous”, but have a hole in them. The holes are called removable discontinuities because it is possible to obtain a continuous function by filling them in with a point. It is possible to take the limit of a function as \( x \) approaches a removable discontinuity, and this is done by pretending the hole is not there and evaluating the function at the point anyway, much as we did with the previous 5 examples. Examples:

(a) \( f : \mathbb{R} \setminus \{0\} \to \mathbb{R}, f(x) = \frac{x^2}{|x|} \) (Plot this. Why is there a hole at \( x = 0 \)?) Here we have \( \lim_{x \to 0} f(x) = 0 \), even though \( f(0) \) is not defined.

**Note:** The notation \( \mathbb{R} \setminus \{0\} \) means “the set of real numbers, except for the point \( x = 0 \)”. Essentially it allows us to throw away single points where a function would not be defined otherwise. In general \( \mathbb{R} \setminus \{a_1, a_2, \ldots, a_n\} \) is the set of real numbers, except for the points \( x = a_1, x = a_2, \ldots, x = a_n \).

(b) \( g : \mathbb{R} \setminus \{0\} \to \mathbb{R}, g(x) = \frac{x^3}{x} \). \( \lim_{x \to 0} |g(x)| = 0 \) but \( g(0) \) is not defined.

(c) \( h : \mathbb{R} \to \mathbb{R}, \) given by

\[
    h(x) = \begin{cases} 
    x^2 - 3, & x \neq -1 \text{ and } x \neq 1 \\
    2, & x = -1 \\
    3, & x = 1 
    \end{cases}
\]

Here \( h(-1) = 2 \) and \( h(1) = 3 \), but \( x \to -1 \Rightarrow (-1)^2 - 3 = -2 \) and \( x \to 1 \Rightarrow (1)^2 - 3 = -2 \).

**Exercise:** plot the above functions

**Common situations where a limit does not exist**

It is good to have a list of examples of limits which do not exist which we can quickly recognize. Common situations where limits do not exist are:

1. Functions with jumps.
   
   (a) \( f(x) = \frac{x}{|x|}; \lim_{x \to 0} f(x) \) does not exist.
   
   (b) \( g(x) = \begin{cases} 
    -x^2 + 1, & x \leq 2 \\
    x, & x > 3 
    \end{cases}; \lim_{x \to 3} g(x) \) does not exist.
   
   (c) \( h(x) = [x] \) is the floor function. It returns the greatest integer which is less than or equal to \( x \) (an integer is a whole number \( \cdots -3, -2, -1, 0, 1, 2, 3, \ldots \)). If \( n \) is an integer, then \( \lim_{x \to n} [x] \) does not exist.
   
   (d) \( F(x) = [x] \) is the ceiling function. It returns the smallest integer which is greater than or equal to \( x \). If \( n \) is an integer, then \( \lim_{x \to n} [x] \) does not exist.
2. Unbounded functions

(a) \( f(x) = \frac{1}{x}; \lim_{x \to 0} f(x) \) does not exist

(b) \( g(x) = \frac{1}{x^2}; \lim_{x \to 0} g(x) \) does not exist. In this case, we may write \( \lim_{x \to 0} g(x) = \infty \), to describe how the limit does not exist.

(c) \( h(x) = \log(|x|); \lim_{x \to 0} h(x) \) does not exist. In this case, we may write \( \lim_{x \to 0} g(x) = -\infty \), to describe how the limit does not exist.

3. Rapidly oscillating functions.

(a) \( f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x > 0 \\ 0 & x \leq 0 \end{cases} \). Here \( \lim_{x \to 0} f(x) \) does not exist.

(b) \( g(x) = e^{\cos(1/x)} \)

Exercise. Plot the graphs of the above functions near the point where the limit does not exist and explain what they look like.

Techniques for evaluating limits

Here is an algorithm for evaluating the limit \( \lim_{x \to a} f(x) \).

1. Graph the function and label \( x = a \) on the graph. This is to make sure there is no monkey business going on.

2. If you recognize that the graph of the function jumps, is unbounded, or oscillates rapidly as \( x \) approaches \( a \), there is certainly monkey business afoot, and we conclude that the limit does not exist.

3. If there is no monkey business that you can see, there is a chance that the limit exists. Try evaluating \( f \) at \( x = a \). If the expression you get makes sense, (i.e. you don’t divide anything by zero) great! Conclude that the limit exists, and is equal to \( f(a) \).

4. If \( f(a) \) is not defined, there is hope yet. Try the following techniques, which we will explain in detail below:

   (a) factor \( (x - a) \) from the numerator and the denominator (here we are assuming \( f \) is a rational function, which is a quotient of polynomials)

   (b) multiply by conjugate radicals

5. If none of the above work, you are dealing with a really wacky function. Either take your best guess (it probably doesn’t exist), or resort to using the \( \epsilon, \delta \) definition of a limit.

Now we will see some examples where \( f \) is a rational function, evaluating \( f \) at \( x = a \) results in division by zero, and we have to factor \( x - a \) from the numerator and denominator to determine that \( \lim_{x \to a} f(x) \) exists.
Exercise (Factoring 1). Find \( \lim_{x \to 3} \frac{x^2 - 9}{x^2 - x - 6} \)

Solution: First graph the function with your calculator. Even though the function is unbounded near \( x = -2 \), it seems reasonably well behaved near \( x = 3 \), which should lead us to believe that the limit exists. Now try evaluating at \( x = 3 \). We should get \( \frac{0}{0} \), which is nonsense. However, the fact that we got zero in the numerator and the denominator tells us that we should be able to factor \( x - 3 \) out of both of these. Upon factoring, we see that

\[
\frac{x^2 - 9}{x^2 - x - 6} = \frac{(x - 3)(x + 3)}{(x - 3)(x + 2)} = \frac{x + 3}{x + 2},
\]

provided that \( x \neq 3 \). This is not a problem when we take a limit (This goes back to the “0 < |x - a| < \delta” assumption in the definition of a limit). Therefore,

\[
\lim_{x \to 3} \frac{x^2 - 9}{x^2 - x - 6} = \lim_{x \to 3} \frac{x + 3}{x + 2} = \frac{3 + 3}{3 + 2} = \frac{6}{5}.
\]

Exercise (Factoring 2). Find \( \lim_{x \to -5} \frac{2x^2 + 7x - 15}{5x^2 + 32x + 35} \)

Solution: Graphing the function, there appears to be no shady behavior as \( x \) approaches \(-5\), so we conjecture that the limit exists. We try plugging in \( x = -5 \) and get \( \frac{0}{0} \). This tells us that we can factor \( x + 5 \) out of the numerator and the denominator. We then find

\[
\lim_{x \to -5} \frac{2x^2 + 7x - 15}{5x^2 + 32x + 35} = \lim_{x \to -5} \frac{(2x - 3)(x + 5)}{(5x + 7)(x + 5)}
\]

\[
= \lim_{x \to -5} \frac{2x - 3}{5x + 7}
\]

\[
= \frac{13}{18}
\]

Exercise (Factoring 3). Find \( \lim_{x \to -5} \frac{2x^2 - 13x + 15}{3x^2 + 13x - 10} \)

Plot the function. Immediately alarm bells should be going off in your head, because the function is unbounded as \( x \) approaches \(-5\). This means that the limit does not exist.

Let’s see what would happen if we had not caught this with our calculator. First, we would have tried evaluating the function at \( x = -5 \), which gives us \( \frac{130}{9} \). This tells us that we can factor \( x + 5 \) out of the denominator, and not the numerator, and thus the method we used in the previous two exercises fails, as it should.

Sometimes extra work is required to put a function in a form that can be easily factored. The following two examples pose such problems.

Exercise (Factoring 4). Find \( \lim_{x \to 8} \frac{\frac{1}{x} - \frac{1}{8}}{x - 8} \).
**Solution** Graph the function. There is a vertical asymptote at \( x = 0 \), but no weird behavior near \( x = 8 \), where we are taking the limit. Attempting to evaluate the function at \( x = 8 \) does not work, because of the “\( 1/x \)” term in the numerator. To get rid of the “\( 1/x \)” that is causing the issue, we use a trick: we multiply the entire expression by \( \frac{x}{x} \), which we can do because this is equal to 1 as long as \( x \neq 0 \). Since we care about the limiting behavior near \( x = 8 \), which is nowhere close to 0, this is a valid thing to do. After this, we apply factoring as normal:

\[
\lim_{x \to 8} \frac{\frac{1}{x} - \frac{1}{8}}{x - 8} = \lim_{x \to 8} \frac{\frac{1}{x} - \frac{1}{8} \cdot x}{x - 8 \cdot x} = \lim_{x \to 8} \frac{1 - \frac{1}{8}x}{x(x - 8)} = \lim_{x \to 8} \frac{-\frac{1}{8}(x - 8)}{x(x - 8)} = \lim_{x \to 8} -\frac{1}{8x},
\]

provided that \( x \neq 8 \). Evaluating this new expression at \( x = 8 \), we find that the original limit is \(-\frac{1}{64}\).

**Exercise (Factoring 5).** Find \( \lim_{x \to 7} \frac{x - 7}{\frac{x}{7} - \frac{1}{7}} \).

**Solution** Using the same method as above, we find that the limit is \(-49\). (Work through this one yourself and see if you can get the right answer)

Now we examine the technique of multiplying by conjugate radicals. The *conjugate radical* of the expression \( a + \sqrt{b} \) is \( a - \sqrt{b} \). Note that \((a + \sqrt{b})(a - \sqrt{b}) = a^2 - b \) by the difference of squares formula, so multiplying by a conjugate radical gives us a way to “get rid of” square roots when they cause problems. In an equation, we cannot multiply by a conjugate radical whenever we feel like it. However, we can multiply and divide by a conjugate radical, because this is the same as multiplying by 1.

**Exercise (Conjugate radicals 1).** Find \( \lim_{x \to 4} \frac{4 - x}{5 - \sqrt{x^2 + 9}} \).

**Solution:** Graphing the function, we see that nothing is amiss as \( x \) approaches 4. However, if we try to evaluate at \( x = 4 \), we get \( \frac{0}{0} \). It turns out we can multiply and divide by the conjugate radical of \( 5 - \sqrt{x^2 + 9} \) to obtain an equivalent expression in which we can factor
\[ \frac{4 - x}{5 - \sqrt{x^2 + 9}} = \left( \frac{4 - x}{5 - \sqrt{x^2 + 9}} \right) \left( \frac{5 + \sqrt{x^2 + 9}}{5 + \sqrt{x^2 + 9}} \right) \]
\[ = \frac{(4 - x)(5 + \sqrt{x^2 + 9})}{25 - (x^2 + 9)} \]
\[ = \frac{(4 - x)(5 + \sqrt{x^2 + 9})}{16 - x^2} \]
\[ = \frac{(4 - x)(5 + \sqrt{x^2 + 9})}{(4 - x)(4 + x)} \]
\[ = \frac{5 + \sqrt{x^2 + 9}}{4 + x}, \]
provided that \( x \neq 4 \). Therefore,

\[ \lim_{x \to 4} \frac{4 - x}{5 - \sqrt{x^2 + 9}} = \lim_{x \to 4} \frac{5 + \sqrt{x^2 + 9}}{4 + x} \]
\[ = \frac{5 + \sqrt{4^2 + 9}}{4 + 4} \]
\[ = \frac{5}{4} \]

Exercise (Conjugate radicals 2). Find \( \lim_{x \to -1} \sqrt{x^2 + 8} - 3 \)

Solution: The limit exists and is equal to \(-\frac{1}{3}\). See if you can work through this one on your own and get the right answer.

Exercise (Conjugate radicals 3). Find \( \lim_{x \to 4} \frac{4x - x^2}{2 - \sqrt{x}} \)

Solution: The limit exists and is equal to 16.

Limit Theorems

Theorem (Algebraic Properties of limits). Suppose \( f : \mathbb{R} \to \mathbb{R} \) and \( g : \mathbb{R} \to \mathbb{R} \) are functions with \( \lim_{x \to a} f(x) = \ell \) and \( \lim_{x \to a} g(x) = m \). Then

1. \( \lim_{x \to a} (f(x) + g(x)) = \ell + m \),

2. For any constant \( k \in \mathbb{R} \), \( \lim_{x \to a} (kf(x)) = k\ell \),

3. \( \lim_{x \to a} (f(x)g(x)) = \ell m \),
4. \( \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\ell}{m} \), provided that \( m \neq 0 \),

5. If \( h : \mathbb{R} \to \mathbb{R} \) is a continuous function, then \( \lim_{x \to a} h(f(x)) = h(\ell) \).

Typical functions \( h(x) \) which we might see in bullet point 5 are polynomials, square roots, trig functions, exponential functions, and logarithms.

**Exercise (Algebraic Properties of limits).** Suppose that \( f, g, h : \mathbb{R} \to \mathbb{R} \) are functions which satisfy

\[
\begin{align*}
\lim_{x \to 3} f(x) &= 3 \\
\lim_{x \to 3} g(x) &= -2 \\
\lim_{x \to 3} h(x) &= 7.
\end{align*}
\]

Find

1. \( \lim_{x \to 3} f(x) g(x) - 2h(x) \)
2. \( \lim_{x \to 3} \frac{f(x) - 2g(x)}{\log(h(x))} \)
3. \( \lim_{x \to 3} \cos(h(x)) e^{\sqrt{2f(x)+g(x)}} \)
4. \( \lim_{x \to 3} (f(x) + g(x) + h(x))^3 \).

**Solution:**

1. \( \lim_{x \to 3} f(x) g(x) - 2h(x) = 3 \cdot (-2) - 2(7) = -21 \)
2. \( \lim_{x \to 3} \frac{f(x) - 2g(x)}{\log(h(x))} = \frac{3 - 2(-2)}{\log(7)} = \frac{7}{\log(7)} \)
3. \( \lim_{x \to 3} \cos(h(x)) e^{\sqrt{2f(x)+g(x)}} = \cos(7) e^4 \)
4. \( \lim_{x \to 3} (f(x) + g(x) + h(x))^3 = 8^3 = 512. \)

**Exercise (Algebraic Properties of Limits 2).** If \( \lim_{x \to 4} \frac{f(x) - 5}{x - 2} = 1 \), then what is \( \lim_{x \to 4} f(x) \)?

**Solution:** Assume that \( \lim_{x \to 4} f(x) = \ell \). Using the algebraic properties of limits, we see that

\[
1 = \lim_{x \to 4} \frac{f(x) - 5}{x - 2} = \frac{\ell - 5}{2}.
\]

We solve for \( \ell \) and find \( \ell = 7 \).
Exercise (Algebraic Properties of Limits 3). If \( \lim_{x \to -1} e^{f(x)-g(x)} = 2 \) and \( \lim_{x \to -1} 2f(x) + g(x) = 3 \), then what are \( \lim_{x \to -1} f(x) \) and \( \lim_{x \to -1} g(x) \)?

Solution: \( \lim_{x \to -1} f(x) = 1 + \frac{\log(2)}{3} \) and \( \lim_{x \to -1} g(x) = 1 - \frac{2\log(2)}{3} \).

Exercise (Algebraic Properties of Limits 4). Find \( \lim_{x \to 1} \sin \left( \frac{x^2 + (\pi - 2)x - (\pi - 1)}{x^2 + 2x - 3} \right) \)

Solution: First graph the function. There are some rapid oscillations near \( x = -3 \), but everything looks normal near \( x = 1 \), so we conjecture that the limit exists. Since \( \sin \) is a continuous function, we have

\[
\lim_{x \to 1} \sin \left( \frac{x^2 + (\pi - 2)x - (\pi - 1)}{x^2 + 2x - 3} \right) = \sin \left( \lim_{x \to 1} \frac{x^2 + (\pi - 2)x - (\pi - 1)}{x^2 + 2x - 3} \right),
\]

so we can evaluate the inside limit first, and then evaluate the \( \sin \) function at the number we get. Using factoring, we find that

\[
\lim_{x \to 1} \frac{x^2 + (\pi - 2)x - (\pi - 1)}{x^2 + 2x - 3} = \frac{\pi}{4},
\]

so that

\[
\lim_{x \to 1} \sin \left( \frac{x^2 + (\pi - 2)x - (\pi - 1)}{x^2 + 2x - 3} \right) = \sin \left( \frac{\pi}{4} \right) = \frac{1}{\sqrt{2}}.
\]

Theorem (Order limit theorem). Suppose \( f, g : (a, b) \to \mathbb{R} \), and \( c \in (a, b) \) satisfy \( f(x) \leq g(x) \) for every \( x \in (a, c) \cup (c, b) \). Then

\[
\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x).
\]

A direct corollary of the the order limit theorem is the so called “squeeze theorem”.

Theorem (Squeeze Theorem). Suppose \( f, g, h : (a, b) \to \mathbb{R} \) and that \( c \in (a, b) \). If \( f(x) \leq g(x) \leq h(x) \) for every \( x \in (a, c) \cup (c, b) \) and \( \lim_{x \to c} f(x) = \lim_{x \to c} h(x) = \ell \), then \( \lim_{x \to c} g(x) = \ell \).

Exercise (Squeeze Theorem). Use the squeeze theorem to show that \( \lim_{x \to 0} \sin(x) = 0 \) and \( \lim_{x \to 0} \cos(x) = 1 \).

First observe that for all \( x \in \mathbb{R} \), \(-|x| \leq \sin(x) \leq |x|\). Since \( \lim_{x \to 0} |x| = 0 \) and \( \lim_{x \to 0} -|x| = 0 \), the squeeze theorem implies that \( \lim_{x \to 0} \sin(x) = 0 \).

To show that \( \lim_{x \to 0} \cos(x) = 1 \), first note that \( \cos(x) \leq 1 \) for all \( x \in \mathbb{R} \). We obtain a lower bound on \( \cos(x) \) on the interval \((-\pi/4, \pi/4)\) by the function \( h : (-\pi/4, \pi/4) \to \mathbb{R} \) given by

\[
h(x) = \begin{cases} 
\frac{4}{\pi} + 1, & x < 0 \\
-\frac{4}{\pi} + 1 & x \geq 0
\end{cases}
\]

We then have that \( h(x) \leq \cos(x) \leq 1 \) for all \( x \in (-\pi/4, \pi/4) \), and \( \lim_{x \to 0} h(x) = \lim_{x \to 0} 1 = 1 \), so by the squeeze theorem, \( \lim_{x \to 0} \cos(x) = 1 \).
Limits of trigonometric functions

Theorem (special trig limits). The following identities hold:

1. \( \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \)
2. \( \lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = 0 \)

Exercise (trig limits 1). Find \( \lim_{x \to 0} \frac{\sin 3x}{4x} \).

Solution: First we rewrite \( \frac{\sin(3x)}{4x} \) so that the numerator and the denominator are in the same variable \( 3x \):

\[
\frac{\sin 3x}{4x} = \frac{3 \sin 3x}{4 \cdot 3x}.
\]

We make the observation that \( 3x \to 0 \) as \( x \to 0 \). Using the algebraic properties of limits and the above theorem, we have

\[
\lim_{x \to 0} \frac{\sin 3x}{4x} = \frac{3}{4} \lim_{3x \to 0} \frac{\sin 3x}{3x} = \frac{3}{4} \cdot 1 = \frac{3}{4}.
\]

Sometimes we will need to make use of trig identities to find a limit. Common identities are the double angle formulas

\[
\sin 2x = 2 \sin x \cos x
\]

and

\[
\cos(2x) = \cos^2(x) - \sin^2(x) = 1 - 2 \sin^2(x) = 2 \cos^2(x) - 1.
\]

All such identities arise from the amazing Euler identity relating complex exponentials and trig functions:

\[
e^{i\theta} = \cos(\theta) + i \sin(\theta)
\]

(here \( i \) is the imaginary square root of \(-1\)). If you are interested in learning more about this sort of thing, ask me about it.

Exercise (trig limits 2). Find \( \lim_{x \to 0} \frac{\sin x}{\sin 2x} \).

Solution: We utilize the double angle formula:

\[
\lim_{x \to 0} \frac{\sin x}{\sin(2x)} = \lim_{x \to 0} \frac{\sin(x)}{2 \sin(x) \cos(x)} = \lim_{x \to 0} \frac{1}{2 \cos x} = \frac{1}{2}.
\]

Exercise (trig limits 3). Find \( \lim_{x \to 0} 2x \cot x \).
solution: First we do some algebra to put the above expression in a form that “looks like” \( \frac{\sin x}{x} \), because this is something that we know the limit of.

\[
2x \cot(x) = 2x \left( \frac{\cos x}{\sin x} \right)
= 2 \left( \frac{x}{\sin x} \right) \cos x
= 2 \left( \frac{1}{\sin x/x} \right) \cos x.
\]

Now we take the limit:

\[
\lim_{x \to 0} 2x \cot(x) = \lim_{x \to 0} \left[ 2 \left( \frac{1}{\sin x/x} \right) \cos x \right]
= 2 \left( \frac{1}{\lim_{x \to 0} \sin x/x} \right) \lim_{x \to 0} \cos x
= 2 \left( \frac{1}{1} \right) \cdot 1
= 2.
\]

Section 2.4

The limits we have been working with so far describe the behavior of a function \( f \) as \( x \) gets close to \( a \). We do not specify whether \( x \) approaches \( a \) from the left or from the right, however, and must consider both of these possibilities. Sometimes it is advantageous to have the ability to stipulate whether \( x \) approaches \( a \) from the left or the right, and to allow us to do so, we define one sided limits.

It will help to consider an example before defining one sided limits formally. Consider the function \( f : \mathbb{R} \setminus \{0\} \to \mathbb{R} \), given by \( f(x) = \frac{x}{|x|} \). Plotting \( f \), we see that \( f \) assumes the constant value \(-1\) on \((-\infty, 0)\) and then jumps to assume the constant value \(1\) on \((0, \infty)\). The limit of \( f \) as \( x \) approaches zero does not exist, because there is a jump discontinuity here. However, as \( x \) approaches zero from the left, the value of \( f \) approaches \(-1\). We thus say that the limit of \( f \) as \( x \) approaches zero from the left is \(-1\), and write \( \lim_{x \to 0^-} f(x) = -1 \).

As \( x \) approaches zero from the right, the value of \( f \) gets close to \(1\). We say that the limit of \( f \) as \( x \) approaches zero from the right is \(1\), and we denote this by \( \lim_{x \to 0^+} f(x) = 1 \).

The formal definition of one sided limits are very similar to the \( \epsilon/\delta \) definition which we have already encountered for two sided limits.

**Definition (One sided limits).** Let \( f : \mathbb{R} \to \mathbb{R} \) be a function. We say that the limit of \( f \) as \( x \) approaches \( a \) from the right is \( \ell \) if for every \( \epsilon > 0 \), there exists some \( \delta > 0 \) such that, if
$x \in (a, a + \delta)$, then $|f(x) - \ell| < \epsilon$. We then write

$$\lim_{x \to a^+} f(x) = \ell.$$  

We say that the limit of $f$ as $x$ approaches $a$ from the left is $m$ if for every $\epsilon > 0$, there exists some $\delta > 0$ such that, if $x \in (a - \delta, a)$, then $|f(x) - m| < \epsilon$. We then write

$$\lim_{x \to a^-} f(x) = m.$$  

For every theorem about the algebraic and order properties of limits which we encountered in section 2.2, there is an analogous theorem for one sided limits.

**Theorem (Relationship between one sided/two sided limits).** Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then $\lim_{x \to a} f(x) = \ell$ if and only if the one sided limits $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist and are both equal to $\ell$.

**Exercise (One sided limits).** Define

$$f(x) = \begin{cases} 
\sqrt{-x}, & x < 0 \\
n & x = 0 \\
2x^2, & 0 < x < 1 \\
1 - \sqrt{x - 1}, & x \geq 1 
\end{cases}$$

Find $\lim_{x \to 0^+} f(x)$ and $\lim_{x \to 1^-} f(x)$ using one sided limits.

**Solution:** To find $\lim_{x \to 0} f(x)$, note that

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \sqrt{-x} = 0,$$

and that

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 2x^2 = 0.$$  

By the theorem relating one sided limits and two sided limits, $\lim_{x \to 0} f(x)$ exists and is equal to 0.

Now to find $\lim_{x \to 1} f(x)$, note that

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 2x^2 = 2,$$

and that

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 1 - \sqrt{x - 1} = 1.$$  

Since the right and left hand limits are not equal, we conclude that $\lim_{x \to 1} f(x)$ does not exist.
Exercise (One sided limits 2). Plot the functions \( u(x) = \frac{x}{|x|} \) and \( v(x) = \frac{x}{2|x|} + \frac{1}{2} \). Describe what these functions look like. Do \( \lim_{x \to 0} u(x) \) and \( \lim_{x \to 0} v(x) \) exist? Prove your answer using one sided limits.

Solution: First consider \( u(x) \). Recall that if \( x < 0 \), then \( |x| = -x \), and if \( x \geq 0 \), then \( |x| = x \). If we take the left hand limit, then as \( x \) approaches zero, it assumes only negative values, and therefore

\[
\lim_{x \to 0^-} u(x) = \lim_{x \to 0^-} \frac{x}{|x|} = \lim_{x \to 0^-} \frac{x}{-x} = \frac{1}{-1} = -1.
\]

On the other hand, when we take the right hand limit, \( x \) will approach 0 from the right, assuming only positive values, and therefore

\[
\lim_{x \to 0^+} u(x) = \lim_{x \to 0^+} \frac{x}{|x|} = \lim_{x \to 0^+} \frac{x}{x} = \frac{1}{1} = 1.
\]

We conclude that \( \lim_{x \to 0} u(x) \) does not exist. The same approach may be used to show that \( \lim_{x \to 0} v(x) \) does not exist.

Exercise (One Sided Limits). Let \( g \) and \( h \) be defined by

\[
g(x) = \left( \frac{x - \frac{\pi}{2}}{2 |x - \frac{\pi}{2}|} + \frac{1}{2} \right) \sin(x)
\]

and

\[
h(x) = \left( \frac{x - \frac{\pi}{2}}{2 |x - \frac{\pi}{2}|} + \frac{1}{2} \right) \cos(x).
\]

What are \( \lim_{x \to \frac{\pi}{2}} g(x) \) and \( \lim_{x \to \frac{\pi}{2}} h(x) \)?

Solution: Note first that if \( x < \frac{\pi}{2} \), then \( |x - \frac{\pi}{2}| = \frac{\pi}{2} - x \), and if \( x \geq \frac{\pi}{2} \), then \( |x - \frac{\pi}{2}| = x - \frac{\pi}{2} \). Thus

\[
\lim_{x \to \frac{\pi}{2}^-} g(x) = \lim_{x \to \frac{\pi}{2}^-} \left( \frac{x - \frac{\pi}{2}}{2 \left( \frac{\pi}{2} - x \right)} + \frac{1}{2} \right) \sin(x) = 0
\]

and

\[
\lim_{x \to \frac{\pi}{2}^+} g(x) = \lim_{x \to \frac{\pi}{2}^+} \left( \frac{x - \frac{\pi}{2}}{2 \left( x - \frac{\pi}{2} \right)} + \frac{1}{2} \right) \sin(x) = 1 \cdot \sin \left( \frac{\pi}{2} \right) = 1.
\]

Thus \( \lim_{x \to \frac{\pi}{2}} g(x) \) does not exist. Using similar methods, it can be shown that \( \lim_{x \to \frac{\pi}{2}} h(x) \) exists and is equal to zero, because both of the one sided limits are equal to zero.
Exercise (One Sided Limits). For each real number \( a \), define a function \( f_a(x) \) by

\[
f_a(x) = \begin{cases} 
-2x + a, & x < -1 \\
1, & x = -1 \\
x^2 - a, & x > -1
\end{cases}
\]

(This is called a function with a parameter) For which values of \( a \) does \( \lim_{x \to -1} f_a(x) \) exist? Answer the same question for \( \lim_{x \to -1^+} f_a(x) \) and \( \lim_{x \to -1^-} f_a(x) \). What are the values of these limits?

Section 2.5

Definition (Interior Points and End Points). Let \( A \) be a subset of the real numbers. A real number \( x \) is an interior point of \( A \) if there exists some open interval \( I \) containing \( x \) which is a subset of \( A \). We say \( x \) is an end point of \( A \) if any open interval containing \( x \) contains some element of \( A \), as well as some real number which is not an element of \( A \).

Example (Interior Points and End Points). Consider the subset \( A = (1, 2) \cup [3, 4] \) of the real numbers. An example of an interior point of \( A \) is \( x_1 = \frac{3}{2} \), because the open interval \( I_1 = \left( \frac{3}{2}, \frac{7}{2} \right) \) contains \( x_1 \) and is a subset of \( A \). Another example of an interior point of \( A \) is \( x_2 = \frac{13}{4} \), because the open interval \( I_2 = \left( \frac{25}{8}, \frac{27}{8} \right) \) contains \( x_2 \) and is a subset of \( A \). An example of a boundary point of \( A \) is \( x_3 = 4 \), because no matter how small of an open interval containing \( x_3 \) we consider, there will always be a point in this interval which is a little bit smaller than 4, and another point in this interval which is a little bit bigger than 4, and therefore any interval will contain an element of \( A \) and a real number which is not an element of \( A \). Note that it is possible for an endpoint of \( A \) to not be an element of \( A \). For instance, \( x_4 = 2 \) is an endpoint of \( A \), because no matter what interval we choose containing 2, this interval will always contain a point which is a little bit bigger than 2, and another point which is a little bit smaller than 2.

Exercise (Interior points). We saw above that an endpoint of a subset \( A \) of the real numbers could be an element of \( A \), or could possibly be a real number which is not an element of \( A \). Must an interior point of \( A \) be an element of \( A \)?

Definition (Continuity at an Interior Point). Let \( A \) be subset of the real numbers. A function \( f : A \to \mathbb{R} \) is continuous at an interior point \( a \) of \( A \) if all of the following are true:

1. The function value \( f(a) \) exists,
2. the two sided limit \( \lim_{x \to a} f(x) \) exists, and
3. \( \lim_{x \to a} f(x) = f(a) \).

Example (Continuity). Consider the function \( f : \mathbb{R} \to \mathbb{R} \) defined by

\[
f(x) = \begin{cases} 
\frac{1}{x^2}, & x \neq 0 \\
1, & x = 0
\end{cases}
\]
This function is continuous at the interior point \( x = 2 \) of its domain because \( f(2) \) and 
\[ \lim_{x \to 2} f(x) \]
both exist and are both equal to \( \frac{1}{4} \). In fact, \( f \) is continuous at any point in the set 
\( (-\infty, 0) \cup (0, \infty) \), but is not continuous at the point \( x = 0 \) because \( \lim_{x \to 0} f(x) = \infty \).

**Example (Continuity 2).** Consider the function \( g : \mathbb{R} \to \mathbb{R} \) defined by \( g(x) = |x| \), which returns the greatest integer which is less than or equal to \( x \). This function is continuous at the interior point \( x = \frac{3}{2} \) because both \( g(\frac{3}{2}) \) and \( \lim_{x \to \frac{3}{2}} g(x) \) exist and are equal to 1. An interior point of the domain of \( g \) at which \( g \) is not continuous is 2. Here, even though \( g(2) \) exists (and is equal to 2), the limit of \( g(x) \) as \( x \) approaches 2 does not exist because the corresponding one sided limits exist and are not equal. In general, \( g \) is not continuous at any point \( n \) which is an integer (shorthand: \( n \in \mathbb{Z} \)), and is continuous at any real number which is not an integer.

**Exercise (Continuity 3).** Let \( h : \mathbb{R} \to \mathbb{R} \) be given by \( h(x) = \lfloor x^2 \rfloor \). At what real numbers is \( h \) continuous and at which real numbers is \( h \) not continuous?

**Example (Continuity 4).** Let \( F : (-1, 1) \to \mathbb{R} \) be given by
\[
F(x) = \begin{cases} 
1 - x^2, & x \neq 0 \\
2, & x = 0 
\end{cases}
\]
Then \( F \) is continuous at any point in \((-1, 0) \cup (0, 1)\). However, \( F \) is not continuous at 0, because \( f(0) = 2 \), but \( \lim_{x \to 0} f(x) = 1 \).

**Definition (Continuity at an End Point).** Let \( A \) be subset of the real numbers. A function \( f : A \to \mathbb{R} \) is continuous at an end point \( b \) of \( A \) if all of the following are true:

1. The function value \( f(b) \) exists,
2. At least one of the one-sided limits \( \lim_{x \to b^-} f(x) \) or \( \lim_{x \to b^+} f(x) \) exists, and
3. \( f(b) = \lim_{x \to b^-} f(x) \) or \( f(b) = \lim_{x \to b^+} f(x) \).

**Example (Continuity at endpoints).** Let \( f : [-2, 2] \to \mathbb{R} \) be given by \( f(x) = \sqrt{4 - x^2} \). This function is continuous at the endpoints \(-2 \) and \( 2 \) of its domain because \( f(-2), f(2), \lim_{x \to -2^+} f(x) \) and \( \lim_{x \to 2^-} f(x) \) all exist, and \( f(-2) = \lim_{x \to -2^+} f(x) = 0 \) and \( f(2) = \lim_{x \to 2^-} f(x) = 0 \).

**Example (Continuity at endpoints 2).** Let \( g : [0, \infty) \to \mathbb{R} \) be defined by
\[
g(x) = \begin{cases} 
\log(x), & x > 0 \\
0, & x = 0 
\end{cases}
\]
Then \( g \) is not continuous at the endpoint 0 of its domain, because \( \lim_{x \to 0^+} g(x) = -\infty \).
Example (Continuity at endpoints 3). Let \( h : [-1,1) \to \mathbb{R} \) be defined by

\[
h(x) = \begin{cases} 
\frac{1}{2}, & x = -1 \\
1, & -1 < x < 1 
\end{cases}
\]

Then \( h \) is not continuous at the endpoint \(-1\) of its domain because \( h(-1) = \frac{1}{2} \), but \( \lim_{x \to -1^+} h(x) = 1 \). The function \( h \) is also not continuous at the endpoint 1 of its domain because 1 is not an element of the domain.

Definition (Continuity of a function). Let \( A \) be subset of the real numbers. A function \( f : A \to \mathbb{R} \) is \textit{continuous} if it is continuous at all interior points of its domain \( A \), and is continuous at all endpoints of \( A \) which are also elements of \( A \).

Example (Continuous functions). Most of the functions which you are familiar with from pre-calc are continuous. These include polynomials, roots, trig functions, exponentials, logarithms, etc. An example of a function which is not continuous is \( f : (-\infty,0) \cup (0,\infty) \to \mathbb{R}, f(x) = \frac{1}{x} \), because it is not continuous at the endpoint 0 of its domain. Another example of a function which is not continuous is \( g : \mathbb{R} \setminus \{3\} \to \mathbb{R}, g(x) = \frac{x^2-x-6}{x-3} \), because it is not continuous at the endpoint 3 of its domain (because \( f(3) \) does not exist). A third example of a function which is not continuous is \( g : \mathbb{R} \to \mathbb{R}, g(x) = \begin{cases} \cos(1/x), & x \neq 0 \\
0, & x = 0 \end{cases} \), because it is not continuous at 0 (the limit of \( g \) as \( x \) approaches zero does not exist).

Theorem (Algebraic Properties of Continuous functions). Suppose \( f \) and \( g \) are continuous at the point \( x = c \) (in any sense). Then all of the following functions are continuous at \( x = c \):

1. \( f + g \)

2. \( k \cdot f \), where \( k \in \mathbb{R} \) is a constant

3. \( f \cdot g \)

4. \( \frac{f}{g} \), provided \( g(c) \neq 0 \)

5. \( f^n \), where \( n \) is a positive integer

6. \( \sqrt[n]{f} \), provided \( f(c) \geq 0 \) when \( n \) is even.

Theorem (Continuity of Composite functions). If \( f \) is a function which is continuous at \( x = c \), and \( g \) is a function which is continuous at \( x = f(c) \), then the composite function \( g \circ f \) is continuous at \( x = c \).
Continuous Extensions

**Definition (Removable discontinuity).** A function $f$ has a removable discontinuity at $x = c$ if $f$ is not continuous at $x = c$, and there exists a function $\tilde{f}$ such that $f(x) = \tilde{f}(x)$ for all $x \neq c$ and $\tilde{f}$ is continuous at $x = c$. The function $\tilde{f}$ is called a continuous extension of $f$.

**Example (Removable discontinuity).** The function $f : \mathbb{R} \setminus \{-1\} \to \mathbb{R}$ given by $f(x) = \frac{x^2 - 1}{x + 1}$ has a removable discontinuity at $x = -1$. A continuous extension of $f$ is the function $\tilde{f} : \mathbb{R} \to \mathbb{R}$, $\tilde{f}(x) = x - 1$.

**Example (Non-removable discontinuities).** The function $g : \mathbb{R} \setminus \{0\}$, $g(x) = \frac{x}{|x|}$ has a discontinuity at $x = 0$ which is not removable. These sorts of discontinuities are called jump discontinuities.

**Example (Non-removable discontinuities 2).** The function $h : \mathbb{R} \setminus \{0\}$ given by

$$h(x) = \begin{cases} \sin(1/x), & x > 0 \\ 0, & x \leq 0 \end{cases}$$

has a discontinuity at $x = 0$ which is not removable. Such discontinuities are called oscillating discontinuities.

Intermediate and Extreme Value theorems

**Definition (Minimum/Maximum Values).** Let $f : A \to \mathbb{R}$ be a function. A real number $m$ in the range of $f$ is a minimum value of $f$ if $m \leq f(x)$ for every $x$ in $A$. Similarly, a real number $M$ in the range of $f$ is a maximum value of $f$ if $M \geq f(x)$ for every $x$ in $A$.

**Theorem (Extreme Value Theorem).** Let $f : [a,b] \to \mathbb{R}$ be a continuous function on a closed interval. Then there exist points $c$ and $d$ in $[a,b]$ such that $f(c)$ is a minimum value of $f$ and $f(d)$ is a maximum value of $f$.

**Theorem (Intermediate Value Theorem).** Let $f : [a,b] \to \mathbb{R}$ be a continuous function on a closed interval. If $y$ is a real number with either $f(a) \leq y \leq f(b)$ or $f(b) \leq y \leq f(a)$, then there exists a real number $x$ in $[a,b]$ such that $y = f(x)$.

**Application: Root finding**

A useful application of the intermediate value theorem is to numerically approximate solutions to equations involving continuous functions. For instance, suppose we want to approximate a solution to the equation

$$0 = 5x^3 - 3x - 1.$$ 

By looking at a graph of the polynomial $p(x) = 5x^3 - 3x - 1$, we conjecture that the solution has exactly one real solution near $x = 1$. We then pick an interval $I_1$ which we think contains
the solution. In this case we might choose $I_1 = [0, 2]$. We then evaluate the function $p$ at the endpoints and the midpoint of $I_1$:

\[
p(0) = -1 \\
p(1) = 1 \\
p(2) = 33.
\]

Since $p(0) < 0$ and $p(1) > 0$, and $p$ is continuous, the intermediate value theorem allows us to conclude that $p(x) = 0$ for some $x$ in the interval $I_2 := [0, 1]$. We have now narrowed the interval in which our solution must live. We can keep repeating this process, and evaluate $p$ at the endpoints and midpoint of $I_2$:

\[
p(0) = -1 \\
p(1/2) = -15/8 \\
p(1) = 1.
\]

Since $p(1/2) < 0$ and $p(1) > 0$, the intermediate value theorem allows us to conclude that the solution to our original equation is somewhere in the interval $I_3 := [1/2, 1]$. Notice that each time we apply this procedure, the interval in which we are sure the solution must live gets cut in half. Thus if we apply the algorithm $n$ times, we get an estimate for the solution that is accurate to within $\frac{1}{2^n}$ times the width of our initial interval $I_1$.

**Application: Fixed Points**

**Definition (Fixed Point).** Let $f : [0, 1] \to [0, 1]$ be a function. A real number $p$ in $[0, 1]$ is a fixed point of $f$ if $f(p) = p$.

**Theorem (Fixed Point Theorem).** If $f : [0, 1] \to [0, 1]$ is continuous, then $f$ has a fixed point.

**Proof.** Assume $f : [0, 1] \to [0, 1]$ is continuous. If $f(0) = 0$ or $f(1) = 1$, then $f$ has a fixed point and the proof is complete. Thus assume that $f(0) > 0$ and $f(1) < 1$. Define a new function $g : [0, 1] \to [-1, 1]$ by $g(x) = f(x) - x$. Note that $g$ is continuous by the algebraic properties of continuous functions theorem. Since $g(0) = f(0) > 0$ and $g(1) = f(1) - 1 < 0$, we may conclude by the intermediate value theorem that there exists some point $p$ in $[0, 1]$ such that $g(p) = 0$. But then it follows that $0 = g(p) = f(p) - p$, and so $f(p) = p$. We conclude that $p$ is a fixed point of $f$. \[\square\]

**Infinite Limits**

We have already encountered limits which go to infinity or negative infinity as $x$ approaches a point $a$. For instance, the function $f(x) = \frac{1}{x^2}$ goes to infinity as $x$ approaches zero, and the function $g(x) = -\frac{1}{x^2}$ goes to negative infinity as $x$ approaches zero. Note that this does
not mean that these limits exist! Rather, it gives us a qualitative way to describe how these limits do not exist. Until now, we have adopted a heuristic understanding for what it means for a limit to go to infinity or negative infinity. In this section we aim to more precisely define this concept.

**Definition (Infinite Limits as \( x \) approaches a point).** We say that the limit of \( f \) as \( x \) approaches \( a \) is \( \infty \), and write \( \lim_{x \to a} f(x) = \infty \) if for every \( B > 0 \), there exists some \( \delta > 0 \) such that \( f(x) > B \) for every \( x \) satisfying \( |x - a| < \delta \).

We say that the limit of \( f \) as \( x \) approaches \( a \) is \( -\infty \), and write \( \lim_{x \to a} f(x) = -\infty \) if for every \( B > 0 \), there exists some \( \delta > 0 \) such that \( f(x) < B \) for every \( x \) satisfying \( |x - a| \).

We can also define a one sided limit that approaches infinity as \( x \) approaches \( a \) from the left or right. For example, consider the function \( f(x) = \frac{1}{x} \). Then \( \lim_{x \to 0^-} f(x) = -\infty \) and \( \lim_{x \to 0^+} f(x) = \infty \).

**Definition (Vertical Asymptotes).** The line \( x = a \) is a vertical asymptote of the function \( f \) if either

1. \( \lim_{x \to a^-} f(x) = \pm \infty \), or
2. \( \lim_{x \to a^+} f(x) = \pm \infty \).

**Definition (Finite Limits as \( x \) approaches infinity).** We say that the limit of \( f \) as \( x \) approaches positive infinity is \( \ell \), and write \( \lim_{x \to \infty} f(x) = \ell \), if for every \( \epsilon > 0 \), there exists some \( N > 0 \) such that \( |f(x) - \ell| < \epsilon \) for every \( x > N \).

We say that the limit of \( f \) as \( x \) approaches negative infinity is \( m \), and write \( \lim_{x \to -\infty} f(x) = m \), if for every \( \epsilon > 0 \), there exists some \( N > 0 \) such that \( |f(x) - m| < \epsilon \) for every \( x < -N \).

**Example (Finding finite limits as \( x \) approaches infinity).** Find the limit as \( x \) approaches infinity of the function

\[ f(x) = \frac{1}{1 + x^2}. \]

**Solution:** The trick here is to divided everything by the largest power of \( x \) that we see. This amounts to multiplying the function by one, which we are allowed to do.

\[
\lim_{x \to \infty} \frac{1}{1 + x^2} = \lim_{x \to \infty} \frac{1}{1 + x^2} \left( \frac{1/x^2}{1/x^2} \right) = \lim_{x \to \infty} \frac{1/x^2}{1/x^2 + 1} = \lim_{x \to \infty} \frac{1/x^2}{1/x^2 + 1} = \frac{0}{0 + 1} = 0.
\]
Exercise F. Find the limit as $x$ approaches negative infinity of the function

$$g(x) = \frac{x^2}{x^2 + 1}.$$ 

Solution: The answer is 1. See if you can apply the techniques in the previous exercise to arrive at this answer.

Example. Find $\lim_{x \to \infty} x - \sqrt{x^2 - 4}$.

Solution: The trick here is to multiply and divide by the conjugate radical:

$$\lim_{x \to \infty} x - \sqrt{x^2 - 4} = \lim_{x \to \infty} (x - \sqrt{x^2 - 4}) \left( \frac{x + \sqrt{x^2 - 4}}{x + \sqrt{x^2 - 4}} \right)$$

$$= \lim_{x \to \infty} \frac{4}{x + \sqrt{x^2 - 4}}$$

$$= \lim_{x \to \infty} \frac{4/x}{1 + \sqrt{1 - 4/x^2}}$$

$$= 0$$

$$= 0.$$

Exercise. Find

$$\lim_{x \to \infty} \sqrt{16 + 4x^2 - 2x}.$$ 

Solution The limit is 0.

Definition Horizontal Asymptotes. A line $y = \ell$ is a horizontal asymptote of the function $f$ if either

1. $\lim_{x \to \infty} f(x) = \ell$, or

2. $\lim_{x \to -\infty} f(x) = \ell$.

Example Horizontal Asymptotes. Consider the function

$$f(x) = \frac{x}{|x| + 1}.$$ 

Observe that $f$ has two horizontal asymptotes: one at $y = 1$, and another at $y = -1$. This is because (as you should verify) the limit as $x$ approaches infinity of $f$ is 1, while the limit as $x$ approaches negative infinity of $f$ is $-1$.

Definition Infinite limits as $x$ approaches infinity. We say that the limit as $x$ approaches infinity of $f$ is $\infty$ if for every $B > 0$, there exists some $N > 0$ such that $f(x) > B$ for every $x > N$. 

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We say that the limit as \( x \) approaches infinity of \( f \) is \(-\infty\) if for every \( B > 0 \), there exists some \( N > 0 \) such that \( f(x) < -B \) for every \( x > N \).

We say that the limit as \( x \) approaches negative infinity of \( f \) is \( \infty \) if for every \( B > 0 \), there exists some \( N > 0 \) such that \( f(x) > B \) for every \( x < -N \).

We say that the limit as \( x \) approaches negative infinity of \( f \) is \(-\infty\) if for every \( B > 0 \), there exists some \( N > 0 \) such that \( f(x) < -B \) for every \( x < -N \).

We can say even more about how a function goes to infinity by looking at Oblique Asymptotes of functions. This only works when the numerator of a rational function \( \frac{p(x)}{q(x)} \) has degree (as a polynomial) which is one greater then the degree of the denominator.

**Example (Oblique Asymptotes 1).** The line \( \ell(x) = x \) is an oblique asymptote for the function \( f(x) = \frac{x^2 - 4}{x - 1} \). To see this, note that

\[
\lim_{x \to \infty} \frac{f(x)}{\ell(x)} = \lim_{x \to \infty} \frac{x^2 - 4}{x^2 - x} = \lim_{x \to \infty} \frac{1 - \frac{4}{x}}{1 - \frac{1}{x}} = 1.
\]

**Exercise (Oblique Asymptotes 2).** Find an oblique asymptote of the function

\[ g(x) = \frac{2x^3 - x}{x^2 + 3x}. \]

**Example (Oblique asymptotes 3).** Note that it is possible for a function to have more than one oblique asymptote. Consider, for instance, the function \( h(x) = |x| \), which has two oblique asymptotes, namely \( \ell_1(x) = x \) and \( \ell_2(x) = -x \).

**Example (Computing oblique asymptotes with polynomial division).** Here is a way to compute an oblique asymptote of the function

\[ f(x) = \frac{x^3 + 2x - 4}{2x^2 - 3x + 1}. \]

First we perform polynomial long division, obtaining:

\[
\frac{x^3 + 2x - 4}{2x^2 - 3x + 1} = \frac{1}{2}x + \frac{3}{4} + \frac{19/4x - 19/4}{2x^2 - 3x + 1}.
\]

Thus,

\[
\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{1}{2}x + \frac{3}{4} + \frac{19/4x - 19/4}{2x^2 - 3x + 1} = \lim_{x \to \infty} \frac{1}{2}x + \frac{3}{4},
\]

since \( \lim_{x \to \infty} \frac{19/4x - 19/4}{2x^2 - 3x + 1} = 0 \). An accurate oblique asymptote of \( f \) is therefore \( \ell(x) = \frac{1}{2}x + \frac{3}{4} \).