1 Absolute and Local Extrema

Definition (Absolute Extrema). A function $f : A \to \mathbb{R}$ has an **absolute maximum** at $x \in A$ if $f(x) \geq f(z)$ for all points $z \in A$. We say that $f$ has a **absolute minimum** at $y \in A$ if $f(y) \leq f(z)$ for all $z \in A$. The numbers $f(x)$ and $f(y)$ are then called the **maximum and minimum values** of $f$, respectively.

Example (Absolute Extrema). The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 1 - x^2$ has an absolute maximum at $x = 0$, and has no absolute minimum. The function $g(x) = (x - 2)^2$ has an absolute minimum at $x = 2$, and has no absolute maximum. The maximum value of $f$ is 1, and the minimum value of $g$ is 0.

Example (Domain matters!). The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x(x-1)(x+1)$ has no absolute maximum and no absolute minimum, but the function $g : [-1, 1] \to \mathbb{R}$ defined by $g(x) = x(x-1)(x+1)$ has an absolute max and an absolute min (graph the functions to verify this).

Example (Open vs. closed intervals). The function $f : (-1, 1) \to \mathbb{R}$ defined by $f(x) = x$ has no absolute maximum and no absolute minimum, because no matter what point $x$ we choose in the interval $(-1, 1)$, there is a point $y \in (-1, 1)$ with $f(x) < f(y)$ and a point $z \in (-1, 1)$ with $f(z) < f(x)$. In contrast, the function $g : [-1, 1] \to \mathbb{R}$ defined by $g(x) = x$ has an absolute minimum at $x = -1$ and an absolute maximum at $x = 1$.

Example (Uniqueness of absolute extrema and extreme values). It is possible for a function to have an absolute max at more than one point in its domain. For example, the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \cos(2\pi x)$ has an absolute maximum at every integer value of $x$, and the function $g : \mathbb{R} \to \mathbb{R}$, $g(x) = 0$ has an absolute maximum and an absolute minimum at every $x \in \mathbb{R}$. However, the maximum value of a function is unique. This is because if $f(x)$ and $f(y)$ are two maximum values of $f$, then $f(x) \leq f(y)$ (because there is an absolute maximum at $y$, and $f(y) \leq f(x)$ (because there is an absolute maximum at $x$). Putting these two inequalities together gives us $f(x) = f(y)$, so the maximum value of $f$ is unique if it exists.
Theorem Extreme Value Theorem. If $f : [a, b] \to \mathbb{R}$ is a continuous function on a closed interval, then $f$ attains a maximum value and a minimum value on $[a, b]$.

Example (Why continuity in EVT?). Consider the function $f : [-1, 1] \to \mathbb{R}$ given by
$$f(x) = \begin{cases} 1 - x^2 & x \neq 0 \\ 0 & x = 0 \end{cases}.$$ Notice that $f$ is defined on a closed interval, but is not continuous, and that $f$ has no absolute maximum value. Thus the assumption of continuity in the EVT is necessary.

Definition (Local Extrema). Let $A \subseteq \mathbb{R}$. A function $f : A \to \mathbb{R}$ has a local maximum at $c \in A$ if there is some open interval $I$ containing $c$ such that the restriction of $f$ to $I$, i.e. the function $f|_I : I \cap A \to \mathbb{R}$ defined by $f|_I(x) = f(x)$ has an absolute maximum at $c$. We say that the function $f$ has a local minimum at $d \in A$ if there is some open interval $J$ containing $d$ such that $f|_J$ has an absolute minimum at $d$.

Example (Local Extrema). The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x(x+1)(x-1)$ has two local extrema, but no absolute extrema. The function $g : \mathbb{R} \to \mathbb{R}$ given by
$$g(x) = \begin{cases} x & x \geq 0 \\ -x & -1 \leq x < 1 \\ x+2 & x < -1 \end{cases}$$ has a local minimum at $x = 0$ and a local maximum at $x = -1$. This function also has no absolute extrema.

Theorem (Absolute extrema are local extrema). If $f : A \to \mathbb{R}$ has an absolute extremum at $x = c$, then $f$ has a local extremum at $x = c$.

Note: The converse of the above theorem is false. If $f$ has a local extremum at $x = c$, then it is generally not true that $f$ has an absolute extremum at $x = c$.

Definition (critical point). Let $A \subseteq \mathbb{R}$, and let $f : A \to \mathbb{R}$. Recall that an interior point of $A$ is a point $a \in A$ such that there exists some open interval $I$ which is a subset of $A$ and which contains $a$. We say that an interior point $c \in A$ is a critical point of $f$ if either

1. $f'(c) = 0$, or
2. $f'(c)$ does not exist.

Theorem I. If $f : A \to \mathbb{R}$ has a local extremum at an interior point $c \in A$, then $c$ is a critical point of $f$.

Note: The above theorem tells us that $f$ can only have absolute extrema at critical points or endpoints of its domain. This gives us an algorithm to locate the absolute extrema of $f$. It goes as follows:
1. Determine all critical points and endpoints of \( f \).

2. Evaluate \( f \) at all critical points and endpoints, and take the largest and smallest values found.

3. Check for vertical asymptotes of \( f \), what \( f \) does as \( x \) approaches \( \pm \infty \) to determine if the values found in step 2 are absolute extrema.

Note that the above algorithm does not always return absolute extrema, which is why step (3) requires us to check.

**Example (Computing absolute extrema).** Let \( f : \mathbb{R} \to \mathbb{R} \) be given by \( f(x) = x^4 - 3x^3 + 2x^2 \). We find the absolute extrema of \( f \). First, we compute \( f'(x) = 4x^3 - 9x^2 + 4x \). Next, we find the critical points of \( f \), i.e. the interior points \( x \) of \( \mathbb{R} \) at which \( f'(x) \) does not exist or \( f'(x) = 0 \). The derivative is a polynomial, and thus exists at all points in \( \mathbb{R} \). We now solve the equation \( f'(x) = 0 \):

\[
0 = 4x^3 - 9x^2 + 4x = x(4x^2 - 9x + 4),
\]

thus \( x = 0 \) or \( x = \frac{9}{8} \pm \frac{1}{8} \sqrt{17} \). Evaluating these points in \( f \), we see that

\[
\begin{align*}
    f(0) &= 0 \\
    f \left( \frac{9}{8} - \frac{1}{8} \sqrt{17} \right) &\approx 0.207 \\
    f \left( \frac{9}{8} + \frac{1}{8} \sqrt{17} \right) &\approx -0.62.
\end{align*}
\]

Now we check for vertical asymptotes and behavior at \( \pm \infty \). Since \( f \) is a polynomial, it has no vertical asymptotes. Since \( f \) approaches infinity as \( x \) goes to positive infinity or negative infinity, \( f \) has no maximum. However, \( f \) does have an absolute minimum value of \(-0.62\) at \( x = \frac{9}{8} + \frac{1}{8} \sqrt{17} \).

**Exercise (Computing absolute extrema 2).** Let \( g : [-2, 1/2] \setminus \{0\} \to \mathbb{R} \) be defined by \( g(x) = x^2 - 2x + \frac{1}{x^2} \). Find the absolute extrema of \( g \).

**Exercise (Computing absolute extrema 3).** Let \( h : \mathbb{R} \to \mathbb{R} \) be defined by \( h(x) = -(x^3 - 4x)(x^2 - 2x + 5) \). Find the absolute extrema of \( h \).

**The Mean Value Theorem**

Earlier in the semester, we discussed the intermediate value theorem, which said that if a function \( f \) is continuous on an interval \([a, b]\), and \( c \) is a number between \( f(a) \) and \( f(b) \), then we are guaranteed the existence of a solution to the equation \( f(x) = c \) on the interval \([a, b]\).

In this section, we see that differentiable functions have a similar property. Namely, if \( f \) is differentiable on \([a, b]\), then we are guaranteed the existence of a solution to the equation

\[
f'(x) = \frac{f(b) - f(a)}{b - a}
\]
on the interval \([a, b]\). Geometrically, this equation says that there is a point \(x \in [a, b]\) such that the slope of the tangent line at \(x\) is the same as the slope of the secant line to \(f\) between \(a\) and \(b\). Consider the following graph of a function \(f\):

The blue line is the secant line to \(f\) between \(x = a\) and \(x = b\), and intuitively, the graph has to “turn around” at some point between \(x = a\) and \(x = b\), and it is here that we can find a point \(x = c\) where the tangent line has the same slope as the secant line. This tangent line is shown in red.

Now let’s give the formal statement of the mean value theorem:

**Theorem (MVT).** If \(f : [a, b] \to \mathbb{R}\) is continuous on the closed interval \([a, b]\) and differentiable on the open interval \((a, b)\), then there exists a point \(c \in (a, b)\) such that

\[
f'(c) = \frac{f(b) - f(a)}{b - a}.
\]

Let’s think about why the various assumptions in the mean value theorem are necessary. What goes wrong, for instance, if \(f\) is not assumed to be continuous on \([a, b]\)? The problem that arises here is that there is too much freedom in the slope of the secant line between \(a\) and \(b\). For example, the function \(f : [-1, 1] \to \mathbb{R}\) given by

\[
f(x) = \begin{cases} 
-1 & x = -1 \\
-x & x \neq -1 
\end{cases}
\]

has the following graph:
The secant line to \( f \) between \( x = -1 \) and \( x = 1 \) is shown in red, and clearly has slope zero. However, any tangent line to \( f \) on the interval \((-1, 1)\) has slope \(-1\), so the mean value theorem does not hold. What happens if we remove the assumption that \( f \) is differentiable on the interval \((a, b)\)? The problem now is that the point where the function has to “turn around” may not have a well defined tangent line. As a concrete example, consider the function \( g : [-1, 1] \to \mathbb{R}, g(x) = 2 - |x| \), whose graph looks like:

As you can see, the slope of the secant line between \( x = -1 \) and \( x = 1 \) is 0, but there is no point between \( x = -1 \) and \( x = 1 \) where the slope of the tangent line to \( g \) is 0.

Now let’s see why the mean value theorem is true. In order to prove it, we need a result called Rolle’s theorem, which is a special case of the mean value theorem where the slope of the secant line between \( a \) and \( b \) is zero.

**Theorem (Rolle’s Theorem).** Let \( f : [a, b] \to \mathbb{R} \) be a continuous function which is differentiable on \((a, b)\) such that \( f(a) = f(b) \). Then there exists a point \( c \in (a, b) \) satisfying \( f'(c) = 0 \).

The proof of Rolle’s theorem essentially says that “what goes up must come down”. For instance, if we throw a ball in the air and catch it, then there was some instant in time where it turned around in the air. At this instant, the velocity of the ball was zero. The following proof of Rolle’s theorem is a more rigorous version of this statement:

**Proof.** (Rolle’s Theorem) Assume that \( f : [a, b] \to \mathbb{R} \) is continuous, differentiable on \((a, b)\), and satisfies \( f(a) = f(b) \). We consider two cases, when \( f \) is a constant function and when \( f \) is not a constant function. If \( f \) is constant on \([a, b]\), then the derivative of \( f \) is clearly 0, and we are done. Assume instead that \( f \) is not constant on \([a, b]\). Then there is some point \( x \in (a, b) \) such that \( f(x) \neq f(a) \) (and thus \( f(x) \neq f(b) \)). There are two possibilities, either \( f(x) > f(a) \) or \( f(x) < f(a) \). The proof does not vary much between these two cases, so we will only consider the case \( f(x) > f(a) \).

Now, \( f \) is continuous on \([a, b]\), so the extreme value theorem guarantees that \( f \) has an absolute max at some point \( c \) in the interval \([a, b]\). Furthermore, our assumption that \( f(x) > f(a) \) guarantees that \( c \) is not one of the endpoints, and thus \( c \in (a, b) \). Since any absolute max is also a local max, and any local max is a critical point, it follows that \( c \) is a critical point. By the definition of a critical point, either \( f'(c) \) does not exist or \( f'(c) = 0 \). Since \( c \in (a, b) \), and \( f \) is differentiable at every point of \((a, b)\), \( f'(c) \) exists, and therefore \( f'(c) = 0 \). \( \Box \)
We now use Rolle’s theorem to prove the mean value theorem. We do this by taking the function \( f : [a, b] \to \mathbb{R} \), where \( f(a) \) and \( f(b) \) are not necessarily equal, and coming up with a function \( h \) that we can use Rolle’s theorem on. First, we consider the function \( g \) whose graph is the same as the secant line to \( f \) between \( a \) and \( b \). This function is given explicitly by

\[
g(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a).
\]

Notice that \( g'(x) = \frac{f(b) - f(a)}{b - a} \) is just the slope of the secant line. We now consider the function \( h(x) = g(x) - f(x) \), which we can think of as the vertical distance between the graph of \( g \) and the graph of \( f \). Below are the graphs of \( f(x) \), \( g(x) \) and \( h(x) \) in the case where \( f(x) = x^3 - x^2 - 2x \).

The crucial observation to make here is that the tangent line to \( h \) has slope zero at the exact same values of \( x \) where the tangent line to \( f \) has slope \( \frac{f(b) - f(a)}{b - a} \). Thus if we can use Rolle’s theorem to find a point \( c \) where \( h'(c) = 0 \), we get for free that \( f'(c) = \frac{f(b) - f(a)}{b - a} \), which proves the mean value theorem.

**Exercise (A specific example).** Let \( f(x) = x^3 - x^2 - 2x \), and let \( a = -1 \) and \( b = \frac{-1}{4} \). Compute the functions \( g(x) \) and \( h(x) \) above and use Rolle’s theorem to show that there is some point \( c \in (-1, -1/4) \) where \( h'(c) = 0 \).

Now we use the above idea to prove the mean value theorem.

**Proof.** (MVT) Let \( f : [a, b] \to \mathbb{R} \) be continuous and differentiable on \((a, b)\). Define the function \( g : [a, b] \to \mathbb{R} \) by

\[
g(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a),
\]
and define the function \( h : [a, b] \to \mathbb{R} \) by
\[
h(x) = g(x) - f(x).
\]
Since \( f \) and \( g \) are both continuous on \([a, b]\) and are both differentiable on \((a, b)\), it follows from the algebraic properties of continuity and differentiability that \( h \) is continuous on \([a, b]\) and differentiable on \((a, b)\). Furthermore,
\[
h(a) = f(a) + \frac{f(b) - f(a)}{b - a}(a - a) - f(a) = 0,
\]
and
\[
h(b) = f(a) + \frac{f(b) - f(a)}{b - a}(b - a) - f(b) = (f(b) - f(a)) - (f(b) - f(a)) = 0,
\]
so the hypothesis \( h(a) = h(b) = 0 \) of Rolle’s theorem is met. We apply Rolle’s theorem to obtain a point \( c \) such that \( h'(c) = 0 \). But
\[
h'(x) = g'(x) - f'(x)
\]
\[
= \frac{f(b) - f(a)}{b - a} - f'(x),
\]
and therefore
\[
0 = h'(c) = \frac{f(b) - f(a)}{b - a} - f'(c).
\]
Adding \( f'(c) \) to both sides of this equality, we obtain
\[
f'(c) = \frac{f(b) - f(a)}{b - a},
\]
and the proof is complete. \( \square \)

The MVT is remarkably useful because it relates the value of \( f \) to the value of its derivative. This allows us to deduce properties of the function \( f \), given some information about \( f' \).

**Example (Functions with zero derivative are constant).** If \( f'(x) = 0 \) for all points \( x \) in an open interval \((a, b)\), then \( f \) is constant on \((a, b)\).

*Proof.* Assume \( f'(x) = 0 \) for all \( x \in (a, b) \). To show that \( f \) is constant on \((a, b)\), it is enough to show that \( f(x_1) = f(x_2) \) for any distinct points \( x_1 \) and \( x_2 \) in \((a, b)\).

Take \( x_1 \neq x_2 \) in \((a, b)\). Either \( x_1 < x_2 \) or \( x_2 < x_1 \), but the proof in either case is similar, so assume without loss of generality that \( x_1 < x_2 \). Since \( f \) is continuous on \([x_1, x_2]\) and differentiable on \((x_1, x_2)\). The MVT guarantees the existence of a point \( c \in (a, b) \) such that
\[
f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.
\]
But \( c \in (a, b) \), so \( f'(c) = 0 \), thus

\[
0 = \frac{f(x_2) - f(x_1)}{x_2 - x_1},
\]

and since \( x_2 - x_1 \) is nonzero, we have

\[
0 = f(x_2) - f(x_1),
\]

which means \( f(x_1) = f(x_2) \). Since we could have done the same thing for any distinct points \( x_1 \) and \( x_2 \) in \( (a, b) \), we conclude that \( f \) is constant.

\[\square\]

**Example (Functions with the same derivative differ by a constant).** If \( f'(x) = g'(x) \) at each \( x \in (a, b) \), then \( f(x) = g(x) + C \) for some constant \( C \).

**Proof.** If \( f'(x) = g'(x) \) on \((a, b)\), then \((f - g)'(x) = 0\) on \((a, b)\), so by the previous example, \( f - g \) is constant, that is, \( f(x) - g(x) = C \) for some constant \( C \in \mathbb{R} \). We conclude that \( f(x) = g(x) + C \).

\[\square\]

**Exercise (Functions with positive derivative are increasing).** Prove that if \( f'(x) \geq 0 \) for all \( x \in (a, b) \) then \( f \) is increasing, and that if \( f'(x) \leq 0 \) for all \( x \in (a, b) \), then \( f \) is decreasing.

**Hint:** Mimic the approach used to show that functions with zero derivative are constant.

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**The First Derivative Test**

Derivatives give us qualitative information about the behavior of a function on an interval. First derivatives tell us whether a function is increasing, decreasing, or constant on an interval, and second derivatives tell us the concavity of a function on an interval. In this section, we focus on the information obtained from first derivatives.

Recall that a function \( f : \mathbb{R} \to \mathbb{R} \) is *increasing* on an interval \((a, b)\) if \( f(x_1) \leq f(x_2) \) for all points \( x_1 \) and \( x_2 \) in \((a, b)\) with \( x_1 < x_2 \). The function \( f \) is *decreasing* on \((a, b)\) if \( f(x_1) \geq f(x_2) \) for all \( x_1 < x_2 \) in \((a, b)\). If \( f(x_1) < f(x_2) \) for all points \( x_1 \) and \( x_2 \) in \((a, b)\) with \( x_1 < x_2 \), then we say that \( f \) is *strictly increasing* on \((a, b)\).

**Exercise 3.1.** Formulate a definition for what it means for a function \( f \) to be *strictly decreasing* on an interval \((a, b)\).

For example, consider the function \( f(x) = x^3 - 12x - 5 \), defined on all of \( \mathbb{R} \). The graph of \( f \) looks like:
As you can see from the graph, \( f \) is strictly increasing on the interval \((-\infty, -2)\), strictly decreasing on the interval \((-2, 2)\), and is strictly increasing on \((2, \infty)\).

Suppose we were not given a graph of \( f \) though, or even a graphing calculator on which to find a graph of \( f \). What would we do to determine where \( f \) was increasing and decreasing?

As you may have guessed, the derivative comes to the rescue. Recall the following theorem from last section

**Theorem (First derivatives and function behavior).** Let \( f \) be a differentiable function on the interval \((a, b)\).

1. If \( f'(x) > 0 \) for all \( x \in (a, b) \), then \( f \) is strictly increasing on \((a, b)\).
2. If \( f'(x) < 0 \) for all \( x \in (a, b) \), then \( f \) is strictly decreasing on \((a, b)\).

This theorem follows from the mean value theorem: if we take points \( x_1 < x_2 \) in the interval \((a, b)\), then \( f \) is continuous on \([x_1, x_2]\) and differentiable on \((x_1, x_2)\), so by the MVT, there exists a point \( c \in (a, b) \) such that \( f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \). We rewrite this as

\[
f(x_2) - f(x_1) = f'(c)(x_2 - x_1).
\]

Now, \( x_2 - x_1 \) is always positive, so the sign of \( f(x_2) - f(x_1) \) depends solely on the sign of \( f'(c) \). Under assumption (1) in the theorem, \( f'(c) > 0 \) and thus \( f(x_1) < f(x_2) \), hence \( f \) is strictly increasing on \((a, b)\). If instead \( f'(c) < 0 \) (i.e. assumption (2) is true), then \( f(x_1) > f(x_2) \), and so \( f \) is strictly decreasing on \((a, b)\).

Let’s use our theorem to verify our graphical intuition of the function \( f(x) = x^3 - 12x - 5 \). The derivative of \( f \) is

\[
f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x - 2)(x + 2).
\]

We see that if \( a \in (-\infty, -2) \), then \( f'(a) > 0 \), and therefore \( f \) is strictly increasing on \((-\infty, -2)\). What about if \( a \in (-2, 2)\)? Here we see that \( f'(a) < 0 \), and so \( f \) is strictly
decreasing on \((-2, 2)\). Similarly, if \(a \in (2, \infty)\), then \(f'(a) > 0\), and so \(f\) is strictly increasing on \((2, \infty)\).

A crucial observation about \(f'\) in this example is that it is continuous. This allows us to determine if \(f\) is strictly increasing or decreasing on the intervals \((-\infty, -2), (-2, 2),\) and \((2, \infty)\) merely by checking the sign of \(f'\) at one point in each interval. How can we get away with this? Note that the critical points of \(f\) are precisely \(-2\) and \(2\). Thus they are the only points where \(f'(x) = 0\). Now consider one of the intervals, say \((-\infty, -2)\), and check that \(f'(-3) = 3(-3 - 2)(-3 + 2) = 3(-5)(-1) = 15 > 0\). If \(f'\) were negative at some point \(a \in (-\infty, -2)\), then by the intermediate value theorem, there would have to exist some point \(b\) in between \(a\) and \(-3\) where \(f'(b) = 0\), contradicting that \(-2\) and \(2\) are the only critical points of \(f\). We thus know that \(f'(x) > 0\) for every \(x \in (-\infty, -2)\) just by checking that \(f'\) is positive for one point in this interval.

The following theorem summarizes the above discussion

**Theorem (When \(f'\) is continuous, only need to check at one point).** Assume that \(f\) is a differentiable function on \((a, b)\), that \(f'\) is continuous on \((a, b)\), and that \(f\) has no critical points on \((a, b)\).

1. If \(f'(c) > 0\) for some \(c \in (a, b)\), then \(f\) is strictly increasing on \((a, b)\).
2. If \(f'(c) < 0\) for some \(c \in (a, b)\), then \(f\) is strictly decreasing on \((a, b)\).

**Exercise 3.2.** Let \(g(x) = 2x^3 - 6x^2 + 6x - 32\). Find the critical points of \(g\), as well as the intervals on which \(g\) is strictly increasing and strictly decreasing.

**Solution:** Note that \(g'(x) = 6x^2 - 12x + 6 = 6(x^2 - 2x + 1) = 6(x - 1)^2\). The only critical point of \(g\) is therefore \(x = 1\). To figure out whether \(g\) is increasing or decreasing on \((-\infty, 1)\), observe that \(g'\) is continuous, and that \(g'(0) = 6 > 0\). We therefore conclude that \(g\) is strictly increasing on \((-\infty, 1)\). Now note that \(g'(2) = 6 > 0\), so \(g\) is also strictly increasing on the interval \((1, \infty)\).

**Exercise 3.3.** Let \(h(x) = \frac{x^3 - x}{x^2 + 3}\). Determine the critical points of \(h\), and the intervals on which \(h\) is strictly increasing and decreasing.

**Solution:** The derivative of \(h\) is

\[
h'(x) = \frac{(x^2 + 3)(3x^2 - 1) - (x^3 - x)2x}{(x^2 + 3)^2}.
\]

The critical points of \(h\) are the points \(x\) where \(h'(x) = 0\), which is equivalent to \((x^2 + 3)(3x^2 - 1) - (x^3 - x)2x = 0\). Expanding the left hand side, we obtain

\[x^4 + 10x^2 - 3 = 0.
\]

Now using the quadratic equation on the variable \(x^2\), we have

\[x^2 = \frac{-10 \pm \sqrt{100 + 12}}{2} = -5 \pm 2\sqrt{7},
\]

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hence

\[ x = \pm \sqrt{-5 \pm 2\sqrt{7}}. \]

But only two of the above solutions are real, thus

\[ x = \pm \sqrt{-5 + 2\sqrt{7}} \]

are the critical points of \( h \).

Now we classify the behavior of \( h \) on the intervals \((-\infty, -\sqrt{-5 + 2\sqrt{7}}), (-\sqrt{-5 + 2\sqrt{7}}, \sqrt{-5 + 2\sqrt{7}}), \) and \( (\sqrt{-5 + 2\sqrt{7}}, \infty) \). Note that

\[ h'(10) = h'(-10) = \frac{103 \cdot 299 - 200 \cdot 99}{103^2} > \frac{103 \cdot 200 - 200 \cdot 99}{103^2} = \frac{200(103 - 99)}{103^2} > 0, \]

therefore \( h \) is strictly increasing on \((-\infty, -\sqrt{-5 + 2\sqrt{7}})\) and on \((\sqrt{-5 + 2\sqrt{7}}, \infty)\). Now observe that

\[ h'(0) = 3(-1)/9 = -1/3 < 0, \]

so \( h \) is strictly decreasing on \((-\sqrt{-5 + 2\sqrt{7}}, \sqrt{-5 + 2\sqrt{7}})\).

Now we use the above theorems to develop the first derivative test, which is a method for determining if a critical point is a local minimum or a local maximum.

**Theorem (First Derivative Test).** Suppose \( f : (a, b) \to \mathbb{R} \) is continuous on \((a, b)\) and differentiable at every point in \((a, b)\) other than a critical point \( c \in (a, b) \). Then

1. If \( f'(x) > 0 \) on \((a, c)\) and \( f'(x) < 0 \) on \((c, b)\), then \( f \) has a local maximum at \( x = c \),
2. if \( f'(x) < 0 \) on \((a, c)\) and \( f'(x) > 0 \) on \((c, b)\), then \( f \) has a local minimum at \( x = c \), and
3. If \( f'(x) > 0 \) on both \((a, c)\) and \((c, b)\), or if \( f'(x) < 0 \) on both \((a, c)\) and \((c, b)\), then \( f \) has no local extremum at \( x = c \).

**Exercise (Quadratics).** Let \( f(x) = ax^2 + bx + c \) be defined for all real numbers, where \( a \neq 0 \). Determine conditions on \( a, b, \) and \( c \) which guarantee that \( f \) has a local max or a local min, and find the value of the max, min in each case.