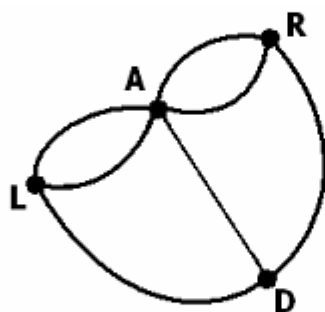


EULER'S THEOREM 1

- If a graph has *any* vertices of odd degree, then it *cannot* have an Euler Circuit.
- If a graph is *connected* and every vertex has even degree, then it has at least one Euler Circuit.

Do we have an Euler Circuit for this problem?



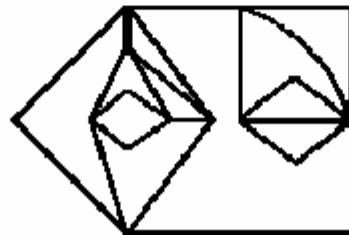
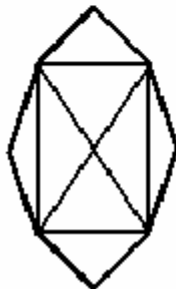
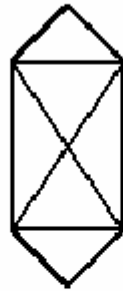
EULER'S THEOREM 2

- If a graph has *more than two* vertices of odd degree, then it *cannot* have an Euler Path.
- If a graph is *connected* and has *exactly two* vertices of odd degree, then it has at least one Euler Path. Any path must start at one of the odd vertices and end at the other.

Is there an Euler Path on the Königsberg problem?

There are 4 vertices and all have odd degree. There cannot be an Euler Circuit and there cannot be an Euler Path. It is impossible to cross all bridges exactly once, regardless of starting and ending points.

UNICURSAL DRAWINGS: Can we trace the drawings below without lifting our pencil or retracing lines? Can we do this starting and ending at the same place?



EULER'S THEOREM 3

- The sum of the degrees of all the vertices of a graph equals twice the number of edges.
- A graph always has an even number of odd vertices.

Summary

# of odd vertices	Conclusion
0	Euler Circuit(s) possible
2	Euler Path(s) possible
4,6,8, ...	Neither
1,3,5, ...	Impossible

Finding an Euler Circuit or Euler Path

Euler's theorems tell us *if a path exists* but not how to find it.

Basic idea for a method: Avoid bridges unless there is no other option. Once we cross a bridge we leave a component of the graph and cannot get back to it.

Important: be organized and clear in which edges you have used. Two copies of the graph can help with this method.

Definition: an algorithm is a set of mechanical rules that, when followed, are guaranteed to produce an answer to a specific problem.

Fleury's Algorithm for finding an Euler Circuit

“Don't cross a bridge until you have to.”

1. First make sure that the graph is connected and all the vertices have an even degree.
2. Start at any vertex.
3. Travel through an edge if a) it is not a bridge for the untraveled part, or b) there is no other alternative.
4. Label the edges in the order in which you travel them.
5. When you can't travel any more, stop.

Note that if we wanted an algorithm for Euler Paths we could use steps 3-5, making sure that we only have two vertices of odd degree and that we start at one and end at the other.

Eulerizing Graphs

Fleury's Algorithm shows us how to find Euler Circuits and Euler Paths, but only on graphs where all vertices are of even degree, or if there are only two vertices of odd degree.

What can we do if there is a graph with odd vertices and we want to find an Euler Circuit?

Example: What do you do if your delivery route doesn't perfectly fit a graph that has an Euler Circuit?

With the paper route, you can't give up or quit, you just repeat an edge. Let's try this with our graphs.

Total Cost of route: = cost of traveling original edges + cost of wasted (deadhead) travel

Solve the problem by adding extra edges to a graph, then apply Fleury's Algorithm.

Eulerizing a graph: the process of changing a graph by adding edges to eliminate the vertices of odd degree.

To minimize the number of additional edges, try to add edges between two odd vertices.

Optimal Eulerization: eulerizing a graph using the fewest possible duplicate edges.

To Eulerize a graph:

1. Identify the odd vertices
2. Add edges to make the odd vertices even
3. An optimal eulerization adds the fewest number of edges possible

Examples:

