Reassessing S3

S3: I can use the limit definition of the derivative to determine the differentiability of a function at a point.

Where topic was first introduced: Modules 5. See also the Desmos activities on piecewise-defined functions and one-sided limits.

Definition

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

Favorite Mistakes:

- Keeping x when finding $f'(a)$. It is so much easier to plug in the value for a before starting any algebra.
- Not evaluating $f(a)$ correctly. $f(a)$ never changes definition, regardless of what h is doing. A function has only one output for any input therefore $f(a)$ cannot change definition. Think of $f(a)$ as the output of the **fixed** point.
- Not using one-sided limits if needed on a piecewise-defined function. When the piecewise-defined function is changing definition at $x = a$, you MUST use one-sided limits, considering $h \to 0^-$ and $h \to 0^+$. To do this appropriately, use the one-sided limit notation. Only after you specify if $h \to 0^-$ or $h \to 0^+$ can you know how to evaluate $f(a+h)$.
- Ignoring absolute values/overreacting to absolute values: If you have an absolute value, wait until the absolute values are impeding your ability to cancel out h. Then you can introduce one-sided limits and reframe $|h|$ as $-h$ for $h < 0$, and $|x|$ as h for $x > 0$.
- Notation, Notation, NOTATION!!!!!

Examples:

1. Let $f(x) = \sqrt{2x+3}$. Is f differentiable when $x = 1$? $f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$
$= \lim_{h\to 0} \frac{\sqrt{2(1+h)+3} - \sqrt{2(1)+3}}{h} = \lim_{h\to 0} \frac{\sqrt{2h+5}-\sqrt{5}}{h} = \lim_{h\to 0} \frac{\sqrt{2h+5}-\sqrt{5}}{h} \frac{\sqrt{2h+5}+\sqrt{5}}{\sqrt{2h+5}+\sqrt{5}} = \lim_{h\to 0} \frac{(2h+5)-5}{h(\sqrt{2h+5}+\sqrt{5})} = \lim_{h\to 0} \frac{2h}{h(\sqrt{2h+5}+\sqrt{5})}$
$\lim_{h\to 0} \frac{2}{\sqrt{2h+5}+\sqrt{5}} = \frac{2}{\sqrt{2(0)+5}+\sqrt{5}} = \frac{2}{\sqrt{5}+\sqrt{5}} = \frac{2}{2\sqrt{5}} = \frac{1}{\sqrt{5}}$

2. Let $f(x) = |x-4|$. Is f differentiable when $x = 4$?

$f'(4) = \lim_{h\to 0} \frac{f(4+h) - f(4)}{h}$ Notice how this is using 4 instead of x
$= \lim_{h\to 0} \frac{|4+h-4| - |4-4|}{h} = \lim_{h\to 0} \frac{|h| - 0}{h} = \lim_{h\to 0} \frac{|h|}{h}$ We evaluated the function at $4+h$ and 4, but we cannot go farther at this point because we do not know if $h > 0$ or if $h < 0$. Introduce $|h|$ as a piecewise-defined function and then use one-sided limits!
Now use $|h| = \begin{cases} h & x \geq 0 \\ -h & x < 0 \end{cases}$ and two one-sided limits:
$\lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = \lim_{h\to 0^+} 1 = 1$
Notice the one-sided limit notation justifying why we can replace $|h|$ with h.

Notice how we simplified BEFORE taking the limit!

$\lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} -1 = -1$

And because $1 \neq -1$, $\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$ does not exist, therefore $f'(0)$ does not exist.

3. Let $g(x) = \begin{cases} x^3+1 & x \le 1 \\ 3x-2 & x > 1 \end{cases}$. Use the definition of derivative to determine if $g'(1)$ exists.

$g'(1) = \lim_{h\to 0} \frac{g(1+h)-g(1)}{h}$ Notice how this is using 1 instead of x

Notice that $g(1)$ is the x coordinate of the FIXED POINT. This value with NOT change for this entire problem. However, $g(1+h)$ WILL change depending on if $h+1>1$ or if $h+1<1$. Thus we introduce one sided limits:

As $h \to 0^+$, $h+1>0$ so $g(1+h) = 3(1+h)-2$. As $h \to 0^-$, $h+1<0$ so $g(1+h) = (h+1)^3+1$ (These are the output values of the FLOATING POINT.) Notice that $g(1)$ is the FIXED POINT and will always be $g(1) = 1^3+1 = 2$.

$\lim_{h\to 0^+} \frac{g(1+h)-g(1)}{h} = \lim_{h\to 0^+} \frac{3(1+h)-2-(1^3+1)}{h} = \lim_{h\to 0^+} \frac{1+3h-2}{h} = \lim_{h\to 0^+} \frac{3h-1}{h} = \lim_{h\to 0^+} \frac{3h}{h} - \frac{1}{h} =$
$\lim_{h\to 0^+} 3 - \frac{1}{h} = -\infty$

Notice this is enough to say the derivative does not exist. However, for the experience, we will continue.

$\lim_{h\to 0^-} \frac{g(1+h)-g(1)}{h} = \lim_{h\to 0^-} \frac{(1+h)^3+1-(1^3+1)}{h} = \lim_{h\to 0^+} \frac{1+3h^2+3h+h^3+1-2}{h}$
$= \lim_{h\to 0^-} \frac{h(3h+3+h^2)}{h} = \lim_{h\to 0^+} 3h+3+h^2 = 3$

And because $3 \ne -\infty$, $\lim_{h\to 0} \frac{g(1+h)-g(1)}{h}$ does not exist, therefore $g'(1)$ does not exist.

Prepare for revision:

First, reflect on your mistake and the correct solution and what you learned: fill in the blanks "I used to think but now I think ________________ because I learned ________________."

Compute the derivative at $x = 0$ from the definition:

1. $f(x) = \frac{|x|}{x}$

2. $f(x) = \begin{cases} -x & x \le 0 \\ x^2 & x > 0 \end{cases}$

3. $g(x) = \begin{cases} x^2+1 & x \le 0 \\ x+1 & x > 0 \end{cases}$

4. $h(x) = \begin{cases} x^2+1 & x \ge 0 \\ -x^2+1 & x < 0 \end{cases}$

(more on next page)

Compute the derivative at the given values.

5. $f(x) = 3x^2 + 9x - 4$. Find $\left.\dfrac{df}{dx}\right|_{x=1}$

6. $g(x) = \sqrt{4x - 1}$. Find $\left.\dfrac{dg}{dx}\right|_{x=3}$

7. $j(x) = \frac{1}{x+2}$. Find $\left.\dfrac{dj}{dx}\right|_{x=4}$

Use the definition of derivative to determine if $g'(1)$ exists.

8. Let $g(x) = \begin{cases} x^3 & x \leq 1 \\ 3x - 2 & x > 1 \end{cases}$.