S2: I can identify limits in indeterminate forms and use appropriate strategies to evaluate the limit.

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Where topic was first introduced: Module 3

Video: `https://www.youtube.com/watch?v=WYZkQ7FGp2Q`

## Background for problems with $\infty$:

1. Basic:

$$\lim_{x\to\infty} x = \infty \qquad \lim_{x\to\infty} 1/x = 0 \qquad \lim_{x\to\infty} e^x = \infty \qquad \lim_{x\to\infty} \ln(x) = \infty$$

$$\lim_{x\to\infty} a^x = \infty, \quad (a > 1) \qquad \lim_{x\to\infty} a^x = 0, \quad (0 < a < 1)$$

2. Rational functions with polynomials: factor out $x$ raised to the highest power from the denominator:

$$\lim_{x\to\infty} \frac{x^4 + x^2 - 9}{4x^3 + 2x - 5} = \lim_{x\to\infty} \frac{\frac{x^4}{x^3} + \frac{x^2}{x^3} - \frac{9}{x^3}}{\frac{4x^3}{x^3} + \frac{2x}{x^3} - \frac{5}{x^3}} = \lim_{x\to\infty} \frac{x + \cancelto{0}{\frac{1}{x}} - \cancelto{0}{\frac{9}{x^3}}}{4 + \cancelto{0}{\frac{2}{x^2}} - \cancelto{0}{\frac{5}{x^3}}} = \infty$$

   Note that the method above is the easiest way to justify your reasoning. Using shortcuts (leading coefficients) would require justification as to why you can ignore the other terms. This can be difficult to do. Please also triple check your limit notation, use of "=" and "$\to$" and algebra.

End behavior:
If $\lim\limits_{x\to\pm\infty} = L$, then $y = L$ is a horizontal asymptote of $f(x)$.
If $\lim\limits_{x\to\pm\infty} = \infty$, then $f(x)$ grows without bound.

To test or check for horizontal asymptotes, take the limit as $x \to \infty$ and if you get a number for that limit, you know you have a horizontal asymptote at $y =$ that number.

Handling parameters: If a problem has a letter like $a$, $b$ or $c$ in it, that letter usually is a parameter and should be treated as a constant. If you are not sure what to you, you could replace the letter with a number to get comfortable with the problem, but for your final answer, you should use the parameter.

$$\lim_{x\to\infty} \frac{3x^2 + ax - 1000}{8x^3 + cx - 5} = \lim_{x\to\infty} \frac{\frac{3x^2}{x^3} + \frac{ax}{x^3} - \frac{1000}{x^3}}{\frac{8x^3}{x^3} + \frac{cx}{x^3} - \frac{5}{x^3}} = \lim_{x\to\infty} \frac{\cancelto{0}{\frac{3}{x}} + \cancelto{0}{\frac{a}{x^2}} - \cancelto{0}{\frac{1000}{x^3}}}{8 + \cancelto{0}{\frac{c}{x^2}} - \cancelto{0}{\frac{5}{x^3}}} = \frac{0}{4} = 0$$

## Favorite Mistakes

1. Evaluate $\lim_{t\to 3} \frac{|t-3|}{t^2-t-6}$ **algebraically**. Show all of your steps and explain all of your work.
   - Evaluate **algebraically** means you need to do an analytic method and use algebra to rewrite the limit until you can evaluate it with substitution. Using a table or graph is a **numeric** solution rather than algebraic solution (Standard L2).

   - Remember that a limit is a value that can be approximated *as accurately as desired...* and this "as desired" piece isn't about one single data point. So saying that $\frac{|3.000001-3|}{(3.000001)^2-(3.000001)-6} = 0.19999996$ means $\lim_{t\to 3^+} \frac{|t-3|}{t^2-t-6} = 0.2$ isn't a valid limit argument because you've chosen only one estimate accuracy- limits only exist if we can do this no matter how accurate we want to be, so we have to do something to show "for all" desired output tolerance.

   - If we try substitution right away, we get $0/0$, but we can't stop here! Start by rewriting the absolute value bits as a piecewise-defined function.

   - Don't skip the one-sided limits. If you got that the limit exists, graph the original function in desmos and see if your result matches the graph. We can't cancel absolute values: $\frac{|t-3|}{t-3} \neq 1$

   - Next, split the limit into two one-sided limits, and use the appropriate piece from the piecewise definition to evaluate the limits, then draw your conclusions.

   - If necessary, factor and cancel any terms but only cancel INSIDE THE CONTEXT OF TAKING A LIMIT: for example:
     $\frac{-(t-3)}{t-3} \neq \frac{-1}{1}$
     BUT
     $\lim_{t\to 3^-} \frac{-(t-3)}{t-3} \neq \lim_{t\to 3^-} \frac{-1}{1}$ because the limit context tells us that $x$ will never be equal to 3 so we won't be dividing by 0.

   - Remember to use limit notation to show us that you are taking a limit.
2. Use algebra to evaluate the limit below.

   $$\lim_{h\to 0} \frac{3\sqrt{1+h+2} - 3\sqrt{3}}{h}$$

   - Not multiplying by the conjugate.
   - Losing the limit notation.
   - Not tracking all the algebra steps carefully creating confusion leading to wrong answers.
   - Using decimals- leave things like $3\sqrt{3}$ as expressions. Students who used decimals didn't have nice cancelations later on because $(3\sqrt{3})^2$ didn't look like $(5.196)^2$.
   - Not taking the limit! Use algebra to get to a point where the $h$ in the numerator factors out to cancel with the $h$ in the denominator, then plug 0 in for $h$.

3. Use algebra to evaluate the limit below.

$$f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{\dfrac{1}{(1+h)^2} - \dfrac{1}{1^2}}{h}$$

- Losing limit notation, not tracking all the algebra steps carefully.
- Working the problem with $x$ instead of 1. This makes the problem significantly more challenging.
- Using magic when the $x$ becomes 1 with no explanation- again, make sure that you use "=" to mean that two expressions are equivalent. An expression with $x$ is not equivalent to a number (and $f'(1)$ is a number.)
- Not taking the limit!

4. We are told that $\lim\limits_{x\to\infty} \dfrac{\sin(x)}{x^2} = 0$. Evaluate $\lim\limits_{x\to\infty} \dfrac{x^2+4}{2x^2-2\sin(x)}$.

$$\lim_{x\to\infty} \frac{x^2+4}{2x^2-2\sin(x)} = \lim_{x\to\infty} \frac{\frac{x^2}{x^2}+\frac{4}{x^2}}{\frac{2x^2}{x^2}-\frac{2\sin(x)}{x^2}} = \lim_{x\to\infty} \frac{1+\cancelto{0}{\frac{4}{x^2}}}{2-\cancelto{0}{\frac{2\sin(x)}{x^2}}} = \frac{1}{2}$$

You must justify why you can get to your answer- taking shortcuts and ignoring the terms you don't want to deal with isn't a justification. Unfortunately, the easiest justification is to just follow the process of dividing everything by $x$ raised to the highest power in the denominator. Also, the leading coefficient shortcut is only good on rational functions where the numerator and denominator are comprised of polynomials. $\sin(x)$ is not a polynomial, so leading coefficients cannot be applied in this context.

## Examples:

1. Evaluate $\lim\limits_{t\to 3} \dfrac{|t-3|}{t-3}$ **algebraically**. Show all of your steps and explain all of your work.
   - First I notice that when $t=3$, the numerator and denominator are both 0, so I'll need to look for a way to factor out and divide by $t=3$. But the absolute value is getting in my way.
   - Rewrite the function as a piecewise-defined function to work around the absolute values:
     $\dfrac{|t-3|}{t-3} = \begin{cases} \frac{-(t-3)}{t-3} & t<3 \\ \frac{t-3}{t-3} & t\geq 3 \end{cases}$.
   - Now evaluate one-sided limits:
     $\lim\limits_{t\to 3^-} \dfrac{|t-3|}{t-3} = \lim\limits_{t\to 3^-} \dfrac{-(t-3)}{t-3} = \lim\limits_{t\to 3^-} \dfrac{-1}{1} = -1$
     $\lim\limits_{t\to 3^+} \dfrac{|t-3|}{t-3} = \lim\limits_{t\to 3^+} \dfrac{(t-3)}{t-3} = \lim\limits_{t\to 3^+} \dfrac{1}{1} = 1$
   - Because $-1 \neq 1$ we conclude that $\lim\limits_{t\to 3} \dfrac{|t-3|}{t-3}$ does not exist

2. $\lim\limits_{h\to 0}\dfrac{3\sqrt{1+h+2}-3\sqrt{3}}{h} = \lim\limits_{h\to 0}\dfrac{3\sqrt{1+h+2}-3\sqrt{3}}{h}\cdot\dfrac{3\sqrt{1+h+2}+3\sqrt{3}}{h(3\sqrt{1+h+2}+3\sqrt{3})} = \lim\limits_{h\to 0}\dfrac{9(h+3)-9(3)}{h(3\sqrt{h+3}+3\sqrt{3})} =$
$\lim\limits_{h\to 0}\dfrac{9h}{h(3\sqrt{h+3}+3\sqrt{3})} = \lim\limits_{h\to 0}\dfrac{9}{3\sqrt{h+3}+3\sqrt{3}} = \dfrac{9}{3\sqrt{0+3}+3\sqrt{3}} = \dfrac{9}{6\sqrt{3}}\ldots$

3. $f(x) = \dfrac{1}{x^2}$. Find $f'(2)$ : $\lim\limits_{h\to 0}\dfrac{f(2+h)-f(2)}{h} = \lim\limits_{h\to 0}\dfrac{\dfrac{1}{(2+h)^2}-\dfrac{1}{2^2}}{h} = \lim\limits_{h\to 0}\dfrac{\dfrac{4-(2+h)^2}{4(2+h)^2}}{h}$
$= \lim\limits_{h\to 0}\dfrac{\dfrac{4-(4+4h+h^2)}{4(2+h)^2}}{h} = \lim\limits_{h\to 0}\dfrac{\dfrac{-4h-h^2}{4(2+h)^2}}{h} = \lim\limits_{h\to 0}\dfrac{\dfrac{h(-4-h)}{4(2+h)^2}}{\frac{h}{1}} = \lim\limits_{h\to 0}\dfrac{h(4+h)}{4(2+h)^2}\dfrac{1}{h} = \lim\limits_{h\to 0}\dfrac{(4+h)}{4(2+h)^2}$
$\dfrac{(4+0)}{4(2+0)^2} = \dfrac{-1}{4}$

4. (Only for Modules 10 and later!)

**L'Hôpital's Rule**

Suppose that $f(a) = g(a) = 0$, and that $f$ and $g$ are differentiable on an open interval $I$ containing $a$ and that $g'(x) \neq 0$ on $I$ if $x \neq a$. Then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

assuming that the limit on the right side of the equation exists.

- You MUST first show that you have an indeterminate limit of either Form $\dfrac{0}{0}$ or Form $\dfrac{\infty}{\infty}$
- You MUST tell us you are using L'Hospital's Rule- either write "L'Hospital" or "LH" or "LR"!

$\lim\limits_{x\to 0}\dfrac{\sin(x)}{x} =$

This is form $\dfrac{0}{0}$ so I'll use L'Hospital's Rule: $\lim\limits_{x\to 0}\dfrac{\sin(x)}{x} = \lim\limits_{x\to 0}\dfrac{\cos(x)}{1} = \dfrac{\cos(0)}{1} = 1$

## Prepare for revision:

First, reflect on your mistake and the correct solution and what you learned: fill in the blanks "I used to think ________________ but now I think ________________ because I learned ________________."

Second, prepare responses to the following questions:

1. $\lim_{h\to 0} \dfrac{h}{h(\sqrt{3+h}+\sqrt{3})} =$

2. $\lim_{h\to 0} \dfrac{9h}{h(3\sqrt{h+3}+3\sqrt{3})} =$

3. $\lim_{h\to 0} \dfrac{(3\sqrt{h+4}-3\sqrt{4})(3\sqrt{h+4}+3\sqrt{4})}{h(3\sqrt{h+4}+3\sqrt{4})} =$

4. $\lim_{h\to 0} \dfrac{\frac{1}{1+h}-1}{h} =$

5. $\lim_{h\to 0} \dfrac{\frac{1}{(1+h)^3}-1}{h} =$

6. Evaluate $\lim_{t\to 3} \dfrac{|t-3|}{t^2-9}$ **algebraically**.

7. Evaluate $\lim_{t\to 3} \dfrac{|t-3|}{(t-3)^2}$ **algebraically**.

8. Evaluate $\lim_{t\to 3} \dfrac{(t-3)^2}{|t-3|}$ **algebraically**.

9. Evaluate $\lim_{t\to 0.5} \dfrac{|4t-2|}{4t-2}$ **algebraically**.

10. $\lim_{x\to\infty} \dfrac{3x^2+5x-42}{8x^3+10x-5}$

11. $\lim_{x\to\infty} \dfrac{3x^3+5x-42}{8x^3+10x-5}$

12. $\lim_{x\to\infty} \dfrac{3x^4+5x-42}{8x^3+10x-5}$

(Only after Module 10)

13. $\lim_{x\to 0} \dfrac{\cos(x)-1}{x}$

14. $\lim_{x\to\infty} \dfrac{e^x}{x^2+2}$