**Revising I2**

I1: Given a non-constant rate of change, I can estimate the total change over a time interval using a Riemann sum.

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Where topic was first introduced: Module 12

## Theory:

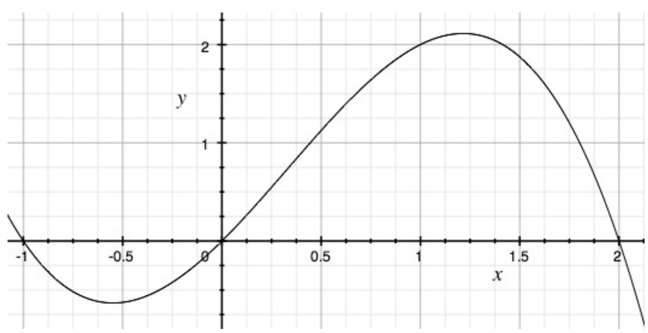
The Riemann definite integral $\int_a^b f(x)dx$ is defined as a limit of Riemann sums. The definite integral can be *approximated* by a finite sum:

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(c_i)\Delta x = f(c_1)\Delta x + f(c_2)\Delta x + \ldots + f(c_{n-1})\Delta x + f(c_n)\Delta x \text{ with } \Delta x = \frac{b-a}{n}$$

Notice that this definition is *different* from the Left and Right Riemann Sums that we have been using.

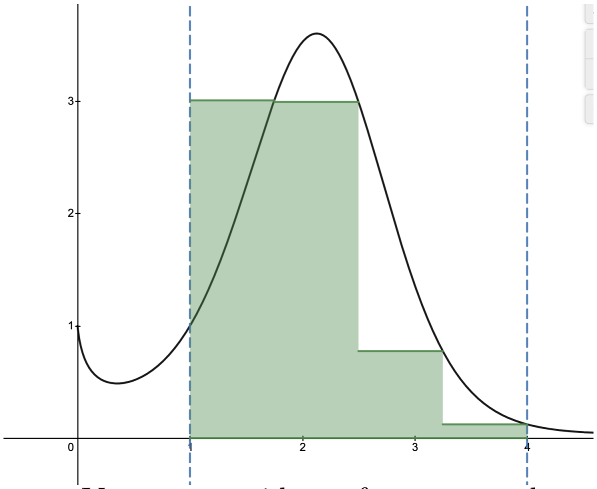
$$\text{Right Sum} = \sum_{i=1}^{n} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_{n-1})\Delta x + f(x_n)\Delta x$$

A function $y = f(x)$ defined on the interval $[a, b] = [-1, 2]$ is shown in the figure.




1. Explain in words how to interpret $x_0, x_1, x_2, \ldots, x_{n-1}, x_n$. (Illustrate in the figure with $n = 6$.)
   The $x$ values are the input values that mark the subdivision of the $x$ axis; they are the endpoints of the subintervals.
2. Explain in words how to interpret $f(c_1), f(c_2), \ldots, f(c_{n-1}), f(c_n)$. (Illustrate in the figure with $n = 6$.)
   The difference between $c_i$ and $x_i$ is that the $x_i$ values are the endpoints of the subintervals where $c_i$ could be any $x$ value on the entire subinterval.
   $f(c_i)$ is the representative height of the function that we will use for that subinterval. Using the the picture, we could think of $f(c_i)$ as the height of the rectangle in a Riemann Sum.
3. The definite integral is defined as a limit of Riemann sums. What does the limit mean, and what purpose does it serve?
   This is asking about the limit in the definition of a Riemann Sum, not limits of integration ($a$ and $b$).

## Example:




- You can use ideas of over or under estimate or where the upper corners of the rectangles touch the curve to decide if the image represents a left or right sum.
- Use the formula $\Delta x = \frac{b-a}{n}$ to get a requested $\Delta x$ by changing $n$.
- $a = 1$ (This is the leftmost $x$ value) and $b = 4$ (rightmost value)
- $n = 4$ because there are 4 rectangles/subintervals.
- $\Delta x = \frac{4-1}{4} = 3/4 = 0.75$
- $x_0 = 1$ (use $a$ for $x_0$ always)
- $x_1 = x_0 + \Delta x = 1.75$
- $x_2 = x_1 + 2 \cdot \Delta x = 2.5$
- $x_3 = x_1 + 2 \cdot \Delta x = 3.25$
- $x_4 = x_1 + 2 \cdot \Delta x = 4$ (Should be the same as $b$)
- Because we are using $x_1, \ldots, x_4$ to get the representative heights, this is a Right Riemann sum: $\displaystyle\sum_{i=1}^{4} f(x_i)\Delta x$
- If we used $x_0, \ldots, x_3$ to get the representative heights, it would be a Left Riemann sum: $\displaystyle\sum_{i=0}^{3} f(x_i)\Delta x$
- $\int_a^b f(x)dx \quad = \displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$ The limit means that we increase the number of subintervals, forcing $\Delta x \to 0$, so as $n \to \infty$ the accuracy of the sum increases until eventually we can no longer measure a difference between the approximation and the actual integral. Thus the limit gives us the exact value of the definite integral.

# Favorite Mistakes:

- Forgetting to multiply by $\Delta x$. (This is how you get the units to work out!)
- Using all the data and having too many intervals. You should only be adding together $n$ items.
- Using the wrong sum. Be sure to use LEFT endpoints for Left sums, RIGHT endpoints for right sums, LARGER outputs for Overestimates/Upper sums and SMALLER outputs for Underestimates/Lower sums.
- Using the antiderivative. (The integrand is already the "how fast" so you can just use that!)

**Examples:**

1. Estimate distance using $n = 4$.

| Time | 4 | 6 | 8 | 10 | 12 |
|---|---|---|---|---|---|
| Speed | 13 | 12 | 10 | 14 | 20 |

   - $\Delta x$ is $\dfrac{12-4}{4} = 2$
   - The LEFT sum is: $13 \cdot 2 + 12 \cdot 2 + 10 \cdot 2 + 14 \cdot 2$
   - The RIGHT sum is: $12 \cdot 2 + 10 \cdot 2 + 14 \cdot 2 + 20 \cdot 2$
   - The UPPER sum is: $13 \cdot 2 + 12 \cdot 2 + 14 \cdot 2 + 20 \cdot 2$
   - The LOWER sum is: $12 \cdot 2 + 10 \cdot 2 + 10 \cdot 2 + 14 \cdot 2$

2. Estimate the value of $\displaystyle\int_2^{10} \frac{x}{3+x}dx$ using 4 subintervals.

   $a = 2$ and $b = 10$ and $n = 4$ so $\Delta x = \dfrac{b-a}{n} = \dfrac{10-2}{4} = 2$

   Then $x_0 = a = 2$, $x_1 = 2 + \Delta x = 2 + 2 = 4$, $x_2 = 6$, $x_3 = 8$, $x_4 = 10 = b$

   - The LEFT sum is: $\dfrac{2}{3+2} \cdot 2 + \dfrac{4}{3+4} \cdot 2 + \dfrac{6}{3+6} \cdot 2 + \dfrac{8}{3+8} \cdot 2$
   - The RIGHT sum is: $\dfrac{4}{3+4} \cdot 2 + \dfrac{6}{3+6} \cdot 2 + \dfrac{8}{3+8} \cdot 2 + \dfrac{10}{3+10} \cdot 2$

# Prepare for Revision:

1. This question is about the approximating the integral: $\displaystyle\int_1^2 \sin(x)\ dx \approx \sum_{i=s}^{q} f(x_i)\Delta x.$

   (a) Explain what is wrong with the following statement:
   The left-hand sum with 10 subdivisions for the integral $\int_1^2 \sin(x)\ dx$ is:

   $$0.1(\sin(1) + \sin(1.1) + \cdots + \sin(2))$$

   (b) What is the role of 0.1? Address why we are multiplying every term by 0.1.

   (c) Link notation to this problem: Rewrite the sum $\displaystyle\sum_{i=s}^{q} f(x_i)\Delta x$, but make it relevant to this problem. You should have numbers instead of $s$ and $q$, and write what the function $f$ is.

   $$\sum_{i=}$$

2. Use a LEFT and a RIGHT sum to estimate $\int_3^7 \ln(x)dx$ with 8 subintervals.

3. Estimate distance using $n = 8$. Use a LEFT sum, a RIGHT sum, an UPPER sum and a LOWER sum.

| | | | | | |
|---|---|---|---|---|---|
| Time | 1 | 2 | 3 | 4 | 5 |
| Speed | 15 | 16 | 10 | 9 | 20 |